REPRESENTATION THEORY – EXERCISES 1

No credits will be awarded for doing the exercises, but I encourage you to do them.

(1) (a) Consider S_3 and its permutation representation $\mathbb{C}[1,2,3]$. In the lectures we used Maschke's theorem to show that $\mathbb{C}[1,2,3] = \mathbb{C}[1+2+3] \oplus \mathbb{C}[2-1, 3-2]$. Prove that $\mathbb{C}[2-1, 3-2]$ is an irreducible S_3-module.
  (b) What are the other irreducible S_3-modules over $\mathbb{C}$?
  (c) Show that, for $n \geq 2$, the symmetric group S_n always has at least two non-isomorphic 1-dimensional representations.

(2) (a) Consider the cyclic group $C_p = \langle a : a^p = 1 \rangle$ for p a prime, and the matrix representation $C_p \to Gl(2, \mathbb{Z}_p)$ sending a^j to the matrix

$$\begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix}$$

Show that the corresponding C_p module $(\mathbb{Z}_p)^2$ has a 1-dimensional C_p-submodule, but cannot be written as a direct sum of 1-dimensional submodules. This shows that the conditions in Maschke's theorem are necessary.
  (b) Using a similar argument, show that the countably infinite cyclic group has a 2-dimensional module over $\mathbb{C}$ which is not completely reducible.

(3) Prove that any finite simple group has an irreducible faithful representation over $\mathbb{C}$. Is the converse true?

(4) Consider the quaternion group Q_8, which can be presented as $\langle a, b : a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$. This has a 2-dimensional matrix representation given by

$$a \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad b \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Show that the corresponding Q_8-module structure on $\mathbb{C}^2$ is irreducible, and calculate all of the irreducible representations of Q_8.

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