REPRESENTATION THEORY – EXERCISES 2

No credits will be awarded for doing the exercises, but I encourage you to do them.

1. Characters

(1) Prove that if χ is an irreducible character and θ an irreducible 1-dimensional character, then $\chi\theta(g) = \chi(g)\theta(g)$ is an irreducible character. (Hint: use that roots of unity x satisfy $x\overline{x} = 1$.)

(2) A certain group of order 12 has six conjugacy classes with representatives $g_1, \ldots, g_6$ where $g_1 = e$. It has two irreducible characters

g	g_1	g_2	g_3	g_4	g_5	g_6
χ_1	1	$-i$	i	1	-1	-1
χ_2	2	0	0	-1	-1	2

Calculate its full character table using results from the course.

(3) Consider the standard S_n-module $\mathbb{C}[1, \ldots, n]$. Find a formula for the value of its associated character χ at a permutation $\pi \in S_n$.

(4) In this question we will calculate the character table of S_4.
- Describe the five conjugacy classes of S_4.
- Describe two distinct 1-dimensional characters.
- By Maschke's theorem there exists a 3-dimensional G-module U for which
$$\mathbb{C}[1, \ldots, 4] = \mathbb{C}[1 + \ldots + 4] \oplus U.$$
Using the previous question, or otherwise, calculate its character and prove that it is irreducible.
- Using results from the course, conclude that there exist two other irreducible characters, of dimensions 3 and 2 respectively.
- Find the other 3-dimensional irreducible character.
- Finally, complete the character table.

2. Maschke's theorem via the projection formula

(1) In *Maschke's theorem*, one starts with a G-module V, and linear map $p : V \to V$. The *key step* is to show that the averaged linear function
$$q(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} p(gv)$$
is a G-module homomorphism.

On the other hand, the *projection formula* used in the course says that given a G-module V the map
$$k(v) = \frac{1}{|G|} \sum_{g \in G} g.v$$
is a projection with image $V^G = \{v \in V : gv = v\}$.

By considering the internal hom G-module $[V, V] = Vect(V, V)$ with its conjugation action described in the course, show how the key step in Maschke's theorem, namely that q is a G-module map, follows from the projection formula.

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3. Tensor products, homs and the trace

(1) In the course we proved that for finite G-modules V and W there exists an isomorphism of G-modules $\theta_{V,W} : [V, W] \cong V^* \otimes W$ indirectly. Describe such an isomorphism directly.

(2) Check that the composite

$$[V, V] \xrightarrow{\theta_{V,V}} V^* \otimes V \xrightarrow{ev} k$$

gives a (matrix free) interpretation of the trace homomorphism. (If it doesn't find an isomorphism so that it does!)