# REPRESENTATION THEORY OF FINITE GROUPS – FOR MD131

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## 0. Introduction

- You will be familiar with group actions on sets $G \to End(X)$. In this course we deal with groups acting linearly on vector spaces $G \to Gl(V)$, which amounts to viewing group elements as matrices/linear transformations.
- Primarily, we will look at complex vector spaces and finite dimensional representations of finite groups – there is a very nice structure theory for these.
- The aim is to have lots of examples – note: in the future, I would like to add more examples and pictures.
- We give a thorough fairly category theoretic treatment of tensor products and homs of representations, and their relationship – using this to understand characters of groups.
- We then classify the irreducible representations of the symmetric group.
- At the end, we touch on Hopf algebras and their representations, and observe that this more general setting captures the nice tensor-hom structure of representations of groups.
- Books that I am using:
  1. *Representations and characters of groups* by *James and Liebeck*. Very gentle book that we used at Warwick when I was there. Lots of examples.
  2. *Representation theory* by *Fulton and Harris* for the approach to the inner product of characters.
  3. *The symmetric group – representations, combinatorial algorithms, and symmetric functions* by *Sagan* for the stuff about the symmetric group.
  4. *Quantum groups: a path to current algebra* by *Street* for the Hopf algebra stuff.

**Remark 0.1.** The students had all taken a course in category theory, so in the end I made the course more category theoretic than I had originally planned.

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## 1. Basics of representation theory

1.1. **Matrix representations.** Let $F$ be a field. By $Gl_d(F)$ we mean the group of invertible $d$-dimensional matrices, with group operation given by matrix multiplication.

**Definition 1.1.** Let $G$ be a group. A $d$-dimensional *matrix representation* of $G$ over $F$ is a homomorphism of groups $\rho : G \to Gl_d(F)$.

In other words, for each $g \in G$ we have an invertible matrix $\rho(g)$, and these satisfy $\rho(g)\rho(h) = \rho(gh)$ and $\rho(1) = I$, the identity matrix.

**Example 1.2.** The *trivial* representation of a group $G \to Gl_1(F)$ sends each element to (1).

**Example 1.3.** Let $C_n = \langle g, g^n = 1\rangle$ be the cyclic group of order $n$. We will calculate all the complex 1-dimensional representations of $C_n$ – that is, homomorphisms $\rho : C_n \to \mathbb{C}$. Letting $\rho(g) = c$ we must have $\rho(g^n) = c^n = 1$ so that $c$ is an $n$-th complex root of unity. There are precisely $n$ of these – $cos(2k\pi/n) = isin(2k\pi/n)$ for $k = 0, n-1$ – and so there are exactly $n$ such representations. For instance, in the case $C_4$ we have $\{1, i, -1, -i\}$ are the roots of unity.

**Example 1.4.** Let $D_8$ be the dihedral group $\langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. This group has 8 elements and describes the symmetries of the square – which are generated by a rotation of order 4 and a reflection.

Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

be the matrices representing such a rotation, and a reflection through the $x$-axis. Now defining $\rho : C_n \to Gl_2(F)$ by $a, b \mapsto A, B$ we obtain a matrix representation of $C_n$ upon checking that $A^4 = B^2 = I$ and $B^{-1}AB = A^{-1}$. Writing down the 8 matrices that you get, we see that no two are the same – equivalently, $ker(\rho) = 1$ – which is to say that $\rho$ is a *faithful representation*.

The following is easy.

**Lemma 1.5.** *If $\rho : G \to Gl_n(F)$ is a representation and $T \in G_n(F)$ then the assignment $G \to Gl_nF : g \mapsto T^{-1}\rho(g)T$ is a representation.*

*Proof.* $T^{-1}\rho(gh)T = T^{-1}\rho(g)\rho(h)T = T^{-1}\rho(g)TT^{-1}\rho(h)T$ and $T^{-1}\rho(1)T = T^{-1}IT = T^{-1}T = I$. □

We can get lots of new examples like this.

**Example 1.6.** Consider $D_8$ again, and its representation from Example 1.4. Choose

$$T = 1/\sqrt{2} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

to diagonalise $A$. We get

$$T^{-1}AT = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad T^{-1}BT = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

This yields another (complex) matrix representation of $D_8$.

**Definition 1.7.** Two matrix representations $\rho, \sigma : G \rightrightarrows Gl_d(F)$ are *equivalent* if there exists $T \in Gl_d(F)$ such that $T^{-1}\rho(g)T = \sigma(g)$ for all $g \in G$.

**Remark 1.8.** It is easy to see that this is an equivalence relation on the set of $d$-dimensional matrix representations over $F$.

1.2. **$G$-modules.** Matrices correspond to linear transformations, up to a choice of basis. For $V$ be a vector space over $F$ let $Gl(V)$ denote the group of invertible linear transformations of $V$.

**Definition 1.9.** Let $G$ be a group. A $G$-module/representation of $G$ is a homomorphism of groups $\rho : G \to Gl(V)$ for some $V$.

Equivalently, a $G$-module is specified by

- a vector space $V$,
- for each $g \in G$ and $v \in V$ an element $\rho(g)(v) \in V$, which we usually just write as $g.v$ or even $gv$, such that:
- $g(v + w) = gv + gw$ and $g(\lambda v) = \lambda(gv)$ for $\lambda \in F$,
- $(gh)v = g(h(v))$ and $1v = v$.

As we know from first year linear algebra, any choice of basis $B$ for a $d$-dimensional vector space $V$ induces an isomorphism of groups $[-]_B : Gl(V) \cong Gl_d(F)$. This sends a linear transformation $X : V \to V$ to the corresponding $d$-dimensional matrix $[X]_B$. Since $[-]_B$ is a group isomorphism, sending a $G$-module $\rho$ to the composite group homomorphism

$$[-]_B \circ \rho : G \to Gl(V) \to Gl_d(F)$$

gives rise to a bijection between $G$-module structures on $V$ and $d$-dimensional matrix representations of $G$. Let us record this.

**Proposition 1.10.** *For a $d$-dimensional vector space $V$, there is a bijection between $G$-module structures on $V$ and $d$-dimensional matrix representations of $G$. Given a $G$-module $\rho$ the corresponding matrix representation $[\rho]_B$ has value $[\rho(g)]_B$ at $g \in G$.*

**Remark 1.11.** Of course $V = k^d$ has a standard basis – we typically view a matrix representation as a representation with respect to this basis via $g.v = \rho(g)(v)$ where $v \in k^d$.

In particular, $G$-modules of finite degree and matrix representations of $G$ are essentially the same thing. As for classical linear algebra, the use of vector spaces rather than just matrices allows a tidier development of the theory, and we will usually work with $G$-modules as opposed to the corresponding matrix representations.

**Definition 1.12.** Let $G$ be a group and $V, W$ be $G$-modules over $F$. A homomorphism of $G$-modules is a linear transformation $\theta : V \to W$ such that $\theta(g.v) = g.\theta(v)$ for all $g \in G, v \in V$. It is an *isomorphism* of $G$-modules if $\theta : V \to W$ is an invertible linear transformation.

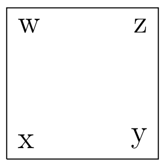
As for other kinds of algebraic structures, we only really care about $G$-modules up to isomorphism. The following exercise expressed the relationship between isomorphism of $G$-modules and equivalence of matrix representations.

**Exercise 1.13.** Let $(V, \rho)$ and $(W, \sigma)$ be $G$-module structures, and $B$ and $C$ bases of $V$ and $W$ of the same cardinality $d$. Then $V$ and $W$ are isomorphic as $G$-modules if and only if the matrix representations $[\rho]_B$ and $[\sigma]_C$ are equivalent.

**Remark 1.14.** It is routine to see that if $\theta : V \to W$ and $\varphi : W \to U$ are homomorphisms of $G$-modules then so is the composite linear transformation $\varphi \circ \theta : V \to U$, and the identity $1 : V \to V$ is certainly a homomorphism of $G$-modules. It follows that $G$-modules and their homomorphisms form a category $Mod_F(G)$, which comes equipped with a forgetful functor $U : Mod_F(G) \to Vect_F$.

**Example 1.15.** Let $G$ be a group acting on a set $X$, and let $FX$ be the free $F$-vector space with basis $X$ – that is, the elements of $FX$ are $F$-linear combinations $\lambda_1 x_1 + \ldots + \lambda_n x_n$ where the $x_i$ are elements of $X$. Thus the elements of $X$ form a natural basis for $FX$. We can define a $G$-module structure on $FX$ by defining the linear transformations $g.- : FX \to FX$ by extending the functions $g.- : X \to X$ linearly; that is, $g(\lambda_1 x_1 + \ldots + \lambda_n x_n) = \lambda_1 g x_1 + \ldots + \lambda_n g x_n$. This is clearly a $G$-module. $G$-modules arising in this way are called *permutation representations.*

**Example 1.16.** For instance, $D_8$ acts on the four corners $\{w, x, y, z\}$ of the square




so that we obtain a $D_8$-module with basis $\{w, x, y, z\}$. Since the rotation $a$ of order 4 permutes these to $x, y, z, w$ the matrix representation $\rho(a) \in Gl_F(4)$

has value

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

**Example 1.17.** The *regular $G$-module* $k[G]$ is obtained, using the above construction, from the left action of $G$ on itself – that is, by $g.h = gh$. In particular $k[G]$ is the vector space with basis $G$. For $g \in G$ and $\lambda_1 g_1 + \ldots + \lambda_n g_n \in k[G]$ we define

$$g(\lambda_1 g_1 + \ldots + \lambda_n g_n) = \lambda_1 g g_1 + \ldots + \lambda_n g g_n.$$

This representation is of central importance, since for $k = \mathbb{C}$, it contains all *irreducible modules* as submodules.

**Example 1.18.** Consider $C_3 = \langle g | g^3 = 1 \rangle$. Its regular representation, with standard basis $\langle g, g^2, g^3 = e \rangle$ , consists of the three matrices

$$\rho(g) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \rho(g^2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \rho(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.3. **Other perspectives – algebras, modules and categories.** The regular $G$-module $k[G]$ has, in fact, more structure than a $k$-vector space – it is a $k$-algebra. A $k$-algebra is a ring $(A, ., 1.+, 0)$ together with an element $\lambda a$ for each $\lambda \in k$ such that

- the abelian group $(A, +, 0)$ is a $k$-vector space;
- the multiplication $a.b$ is bilinear: that is, it is linear on the right $a.(b+c) = a.b+a, c$ and $a.\lambda b = \lambda a.b$ and similarly on the left $(a+b).c = ab + ac$ and $\lambda a.b = \lambda(a.b)$.

A homomorphism of $k$-algebras $\rho : A \to B$ is one that preserves all of the structure.

In the case of $kG$ we have defined the $k$-vector space structure already. The multiplication is given by extending the group multiplication bilinearly, that is:

$$(\Sigma_{g\in G} \lambda_g g).(\Sigma_{h\in G} \theta_h h) = \Sigma_{g,h\in G} \lambda_g \lambda_h (gh)$$

and it has unit $e$.

If $V$ is a $k$-vector space then the set $End(V)$ of (not-necessarily invertible) linear transformations from $V$ to itself is a $k$-algebra, where the multiplication is given by composition of linear transformations.

A *representation* of a $k$-algebra $A$ on $V$ as a homomorphism of $k$-algebras $\rho : A \to End(V)$.

**Proposition 1.19.** *Representations of the $k$-algebra $k[G]$ are in bijection with representations of the group $G$.*

*Proof.* Let $\rho : k[G] \to End(V)$ be a representation. Then for each $g \in G$ we have $\rho(g) \in End(V)$. Note that since $\rho$ preserves multiplication $\rho(g)$ is invertible; thus $\rho(g) \in Gl(V)$ and we have a group representation $\rho|G : G \to Gl(V)$ by restriction. Since the linear map $\rho$ is determined by its value on the basis $G$, we observe that the map sending algebra representations to group representations is injective. To see that it is surjective, let $\rho : G \to Gl(V)$ be a group representation. Then each $\rho(g) \in End(V)$. Since $k[G]$ is the vector space with basis $G$ the function $\rho : G \to End(V)$ extends uniquely to a linear transformation $\rho : k[G] \to End(V)$, defined by $\rho(\lambda_1 g_1 + \ldots + \lambda_n g_n) = \lambda_1 \rho g_1 + \ldots + \lambda_n \rho g_n$. This preserves the $k[G]$-algebra structure since $\rho$ preserves the group structure – and restricting it to a group representation obviously gives back $\rho : G \to Gl(V)$, as required. □

In this way, we see that the study of representation of algebras generalises that of representations of groups. Some of the results developed herein for group representations can be developed for more general algebras, but we will refrain from this generalization, for the sake of simplicity.

**Proposition 1.20.** *Representations of a $k$-algebra $A$ are precisely representations of its underlying ring $UA$ – that is, left $UA$-module structures.*

*Proof.* Consider an $A$-module – i.e. algebra homomorphism $\rho : A \to End(V)$. This is, in particular, a ring homomorphism $\rho : A \to End(V) \to Ab(V, V)$ – thus $V$ is a $UA$-module. On the other hand suppose $V$ is a $UA$-module. That is, we have a ring homomorphism $\theta : A \to Ab(V, V)$. Now we have a ring homomorphism $.1 : k \to UA : \lambda \to \lambda.1$ and the composite ring homomorphism $k \to UA \to Ab(V, V)$ shows that $V$ is a $k$-vector space – with action $\lambda.v = \theta(\lambda.1).v$. In fact, considering $V$ with this vector space structure, the function $\theta : A \to End(V)$ is an algebra homomorphism since it satisfies $\theta(\lambda.a) = \theta((\lambda.1).a) = \theta(\lambda.1)\theta(a) = \lambda.\theta(a)$. The two constructions are inverse. □

1.4. **A categorical point of view.** Given a group $G$ let $BG$ be the category with one object $\bullet$ and $BG(\bullet, \bullet) = G$. Then a functor $\rho : BG \to$ **Set** is precisely a group action – that is, we have a set $\rho(\bullet) = X$ and functions $\rho(g) : X \to X$ for each $g \in G$ satisfying $\rho(g)\rho(h) = \rho(gh)$ and $\rho(e) = id_X$. Indeed the functor category $[BG, \mathbf{Set}]$ is exactly the category of $G$-sets and equivariant maps – the natural transformations capture precisely the equivariance. The regular action of $G$ on itself is the representable $BG(\bullet, -) : BG \to \mathbf{Set}$, which sends $g \in G$ to the function $g.- : G \to G$.

Similarly, a functor $\rho : BG \to \mathbf{Vect}$ from $BG$ to the category of vector spaces is precisely a $G$-representation. Indeed the functor category $[BG, \mathbf{Vect}]$ is exactly the category of $G$-modules – that is, the natural transformations between functors capture the morphisms of $G$-modules.

The forgetful functor $U : \mathbf{Vect} \to \mathbf{Set}$ has a left adjoint $k[-] : \mathbf{Set} \to \mathbf{Vect}$, which sends the set $X$ to the vector space $k[X]$ with basis $X$. In fact, given a group action $\rho : BG \to \mathbf{Set}$ the corresponding *permutation representation* of $G$ is simply the composite functor $k[-] \circ \rho : BG \to \mathbf{Set} \to \mathbf{Vect}$ where $k[-] : \mathbf{Set} \to \mathbf{Vect}$. In fact we get an adjunction $k[-]_* \dashv U_* : [BG, \mathbf{Set}] \leftrightarrows [BG, \mathbf{Vect}]$ by composition in this way.

We can also understand representations of $k$-algebras from the categorical point of view. The tensor product $\otimes$ of $k$-vector spaces equips the category $\mathbf{Vect}$ of $k$-vector spaces admits a symmetric monoidal closed structure, whose unit is $k$. A $k$-algebra $A$ is exactly a monoid in the monoidal category $(\mathbf{Vect}, \otimes, I)$. The left adjoint $F : \mathbf{Set} \to \mathbf{Vect}$ is *strong monoidal* – in particular, $F(X \times Y) \cong FX \otimes FY$. It follows that it takes monoids to monoids – thus a group $G$ is sent to a $k$-algebra $k[G]$. In fact, $k[G]$ inherits *Hopf algebra* structure from the group structure on $G$ – we will come back to this at the end of the course.

## 2. Irreducible $G$-modules

### 2.1. Reducibility and irreducibility.

**Definition 2.1.** Let $V$ be a $G$-module. A vector subspace $W \subseteq V$ is a *submodule* if $v \in W \implies gv \in W$ for all $v \in V, g \in G$.

**Example 2.2.** Each $G$-module $V$ has two standard submodules – $V$ itself and the zero module $\{0\}$. These two submodules are called *trivial*.

**Example 2.3.** Consider the regular $G$-module $k[G]$. If $G$ is finite, so that $G = \{g_1, \ldots, g_n\}$ then these elements form a basis of $k[G]$. Then we can consider the $1-d$ subspace $k[g_1 + \ldots + g_n]$ of $k[G]$ spanned by $g_1 + \ldots g_n$. then

$$g(\lambda(g_1 + \ldots + g_n)) = \lambda(gg_1 + \ldots + gg_n) = \lambda(g_1 + \ldots + g_n)$$

since left multiplication by $g$ permutes the elements of $G$. Thus $k[g_1 + \ldots + g_n]$ is a submodule of $k[G]$.

**Example 2.4.** More generally, if $G$ acts on a finite set $X = \{x_1, \ldots, x_n\}$, then the corresponding permutation representation $k[X]$ has submodule $k[x_1 + \ldots + x_n]$. For instance, the symmetric group $S_n$ acts naturally on $\{1, \ldots, n\}$ so that $k[1 + \ldots + n] \subseteq k[1, \ldots, n]$ is an $S_n$-submodule.

**Definition 2.5.** A $G$-module is *reducible* if it contains a non-trivial submodule. A $G$-module which is non-zero and not reducible is said to be irreducible.

**Remark 2.6.** A matrix representation $\rho : G \to Gl_d(k)$ is said to be *reducible* if the corresponding $G$-module $g.v = \rho(g)v$ is reducible. What does this mean in matrix terms?

It is reducible iff $k^d$ has a $G$-submodule $U < k^d$. The subspace $U = \langle u_1, \ldots u_m \rangle$ extends to a basis $B = \langle u_1, \ldots, u_m, v_1, \ldots, v_l \rangle$ of $k^d$. Let $T$ be the base change matrix from the standard basis to $B$. In this basis the corresponding matrix representation $[g]_B = T^{-1}\rho(g)T$ has the form

$$[g]_B = \left[\begin{array}{c|c} A(g) & B(g) \\ \hline 0 & C(g) \end{array}\right]$$

where each $A(g)$ is of dimension $m < d$.

In this way, we see that a matrix representation is reducible if and only if it is equivalent to matrix representation admitting a block decomposition of the above form.

**Example 2.7.** Consider again the matrix representation of $D_8$ given by the rotation and reflection.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since this is a 2-dimensional representation, a non-trivial submodule must consist of a 1-d subspace $\langle v \rangle$ such that $gv = \lambda v \subseteq \langle v \rangle$ for each $g \in G$ – thus $v$ must be an eigenvector for both $A$ and $B$.

The rotation matrix has no real eigenvectors – it doesn't preserve any straight lines, so the representation is irreducible over $\mathbb{R}$. The reflection $B$ through the $x$-axis has $(1, 0)$ and $(0, 1)$ as eigenvectors.

The rotation $A$ does have complex eigenvalues $+i, -i$, which we can find using the characteristic polynomial, and associated eigenvectors $(1, i)$ and $(1, -i)$. But $A$ and $B$ have no common eigenvectors – thus the representation is also irreducible over $\mathbb{C}$.

Alternatively, consider this as a representation of $\mathbb{C}^2$ given by $a(x, y) = (y, -x)$ and $b(x, y) = (x, -y)$. Then if we have an eigenvector $(\alpha, \beta)$, $b(x, y) = (x, -y)$ forces $\alpha$ or $\beta$ to be 0. But then $a(x, y) = (-y, x)$ forces the other one to be zero – so no 1-d subspace.

2.2. **Kernels, images and direct sums.** The following is routine.

**Proposition 2.8.** *Let $\theta : V \to W$ be a homomorphism of $G$-modules. Then $ker(\theta) \leq V$ and $im(\theta) \leq W$ are $G$-submodules.*

- Let $U, V \leq W$ be subspaces of a vector spaces. The sum $U + V \leq W$ is the subspace $\{u + v : u \in U, v \in V\}$. We say that it is a (internal) *direct sum* and write
$$W = U \oplus V$$
if each element of $W$ is of the form $u + v$ for unique $u \in U, v \in V$. This is equivalent to saying that $U + V = W$ and $U \cap V = \{0\}$.
- If $U, V \leq W$ are $G$-submodules then $U + V$ is a $G$-submodule since $g(u + v) = gu + gv$. In this case, if $W = U \oplus V$, we call it a direct sum of $G$-submodules.

**Definition 2.9.** A projection/idempotent is a homomorphism of vector spaces $p : V \to V$ such that $p^2 = p$.

**Lemma 2.10.** *If $p : V \to V$ is a projection then $V = ker(p) \oplus im(p)$. Each direct sum of subspaces arises from a unique projection in this way.*

*Proof.* Write $v = pv + (pv - v)$. This is unique since if $v \in ker(p) \cap im(p)$ then $pv = 0$, but on the other hand since $v = px$ we have $pv = v$, so that $v = 0$.

Given a direct sum $V = U \oplus W$ the associated projection $p : V \to V$ is defined by $p(u + v) = u$. □

**Corollary 2.11.** *If the $G$-module morphism $p : V \to V$ is a projection, then $V = ker(p) \oplus im(p)$ is a direct sum of $G$-submodules.*

*Proof.* Since $ker(p)$ and $im(p)$ are themselves $G$-submodules this follows from the above. □

**Theorem 2.12** (Maschke's theorem)**.** *Let $G$ be a finite group and $k$ a field such that $char(K)$ does not divide $|G|$ – e.g. if $k$ is infinite. If $V$ is a $G$-module and $U \leq V$ a proper $G$-submodule, then there exists $W \leq V$ with $V = U \oplus W$.*

*Proof.* Firstly, since $U$ is a subspace of $V$ we can choose linearly independent vectors to extend to a basis where $V = U \oplus W_0$ – this corresponds to giving the projection $p : V \to V$ (meaning $p^2 = p$) such that $im(p) = U$ and $ker(p) = W_0$. (Of course, $p$ need not itself be a homomorphism of $G$-modules, unless $W_0$ is a $G$-submodule.)

What we will do is modify $p$ to a homomorphism $q$ of $G$-modules with image $U$ and satisfying $q^2 = q$ – then $ker(q) \leq V$ will a $G$-submodule and $V = im(q) \oplus ker(q) = U \oplus ker(q)$ as required.

For each $g \in G$ we consider the linear map $p_g : v \mapsto g^{-1}p(gv)$ obtained by conjugating $p$ with the action of $g$ and its inverse. As a composite of three linear maps each $p_g$ is linear. We then define

$$q = \frac{1}{|G|}\sum_{g \in G} p_g$$

as the "average of these maps". Let us check that $q$ is a homomorphism.

$$q(hv) = \frac{1}{|G|}\sum_{g \in G} g^{-1}p(g.hv) \tag{2.1}$$

Since each element of $G$ can be written uniquely as $gh^{-1}$ for $g \in G$, this is equally the sum

$$\frac{1}{|G|}\sum_{g \in G}(gh^{-1})^{-1}p((gh^{-1}).hv) = h.\frac{1}{|G|}\sum_{g \in G} g^{-1}p(gv) = hq(v)$$

where we use that $V$ is a $G$-module in the first equation.

Thus $ker(q) \leq V$ is a $G$-submodule of $V$.

For surjectivity of $q$ consider $v \in U$. Then

$$q(u) = \frac{1}{|G|}\sum_{g\in G} g^{-1}p(g.u) = \frac{1}{|G|}\sum_{g\in G} g^{-1}(g.u) = \frac{1}{|G|}\sum_{g\in G} u = \frac{|G|}{|G|}u = u$$

where in the second equality we use that $gu \in U$ and hence is invariant under $p$.

This gives one direction of $im(q) = U$ – for the inclusion $im(q) \subseteq U$ we use that, since $p$ takes its image in $U$, each $g^{-1}p(gv)$ belongs to $U$.

Since $qv \in U$ and $qu = u$ for $u \in U$ we get that $q^2 = q$.

Then by the lemma, we have that $V = ker(q) \oplus im(q) = ker(q) \oplus U$.

□

**Remark 2.13.** The matrix version of Maschke's theorem says that if a matrix representation $\rho : G \to Gl_d(k)$ is equivalent to one with matrices of the form

$$[g]_B = \left[\begin{array}{c|c} A(g) & B(g) \\ \hline 0 & C(g) \end{array}\right]$$

where each $A(g)$ is of fixed dimension $m < d$, then it is equivalent to one with matrices of the form

$$[g]_B = \left[\begin{array}{c|c} A(g) & 0 \\ \hline 0 & B(g) \end{array}\right]$$

for the same dimension $m < d$.

**Exercise 2.14.** If $F = \mathbb{Z}_p$ the cyclic field of characteristic $p$, show that there is a counterexample to the above. Add detail.

**Example 2.15.** Consider the permutation module of the symmetric group $S_3$ acting on $V = \mathbb{C}[v_1, v_2, v_3]$. We have the $S_3$-submodule $U = \langle v_1+v_2+v_3\rangle$. Let us use the formula of Maschke's theorem to find a $S_3$-submodule $W$ with $V = U \oplus W$.

Firstly let $V = U \oplus W_0$ for $W_0 = \langle v_1, v_2\rangle$. Then the corresponding projection $p : V \to V$ onto $U$ sends $v_1$ and $v_2$ to 0 and $v_3$ to $v_1 + v_2 + v_3$.

Observe that $(12)p(12)v_1 = (12)p(v_2) = 0$. Also $(132)p(123)v_1) = 0$, whilst $(23)$ and $e$ also result in 0. On the other hand $(132)$ and $(13)$ result in $v_1+v_2+v_3$. Thus $v_1 \mapsto 2/6(v_1+v_2+v_3) = 1/3(v_1+v_2+v_3)$, whilst it is easy to see that the same is true for $v_2$ and $v_3$ – thus $\theta : v_i \mapsto 2/6(v_1 + v_2 + v_3)$ is the resulting homomorphism. Then $V = U \oplus ker(\theta)$. Then $ker(\theta) = \langle v_1 - v_2, v_3 - v_2\rangle$ and this is the complementary $S_3$-submodule.

## 3. Decomposing $G$-modules

**Theorem 3.1.** *Let $G$ be a finite group and $k$ a field such that $char(k)$ does not divide $|G|$ – e.g. if $k$ is infinite. Then each finite dimensional $G$-module admits a decomposition*

$$V = V_1 \oplus \ldots \oplus V_n$$

*as a direct sum of irreducible $G$-submodules.*

*Proof.* It is by induction on the dimension of the vector space $V$. If $dim(V) = 1$ it is trivial since each 1-module is irreducible. Otherwise suppose it is true for dimensions lower than that of $V$. If $V$ is irreducible are done – otherwise it has a non-trivial submodule $U$, so that by Maschke's theorem $V = U \oplus W$. Then since $dim(U), dim(W) < dim(V)$ we get $V = (U_1 \oplus \ldots U_k) \oplus (V_1 \oplus \ldots \oplus V_l)$ as required. □

**Remark 3.2.** It is possible to prove, at this level of generality, that the above decomposition is unique in the following sense: namely,

- if $V = W_1 \oplus \ldots \oplus W_m$ is a direct sum of irreducible submodules, then $n = m$ and there is a bijection $\rho : n \to n$ such that $V_i \cong W_{\rho i}$ for each $i \in \{1, \ldots, n\}$.

This is an instance of the *Jordan-Holder theorem.* In the case that $k = \mathbb{C}$ we will prove a stronger result about the form of this decomposition, given in Theorem 4.5 below, which implies the above uniqueness.

**Lemma 3.3** (Schur's lemma)**.** *Let $\theta : V \to W$ be a homomorphism of irreducible $G$-modules.*

*(1) Then $\theta = 0$ or $\theta$ is an isomorphism.*
*(2) If $k$ is algebraically closed and $V = W$ of finite dimension then there exists $\lambda \in \mathbb{C}$ such that $\theta(v) = \lambda v$ for all $v \in V$.*

*Proof.* If $\theta \neq 0$ then $ker(\theta) \neq 0$ and $im(\theta) \neq 0$. Then since $V$ is irreducible $ker(\theta) = V$ and since $W$ is irreducible $im(\theta) = W$. Therefore $\theta$ is both injective and surjective: an isomorphism.

For the second part – if $\theta = 0$ it is trivial. Since $k$ is algebraically closed $\theta$ must have an eigenvalue – i.e. the polynomial $det(\theta - \lambda.Id) = 0$ over $k$ must have a solution. Hence there exists $\lambda \neq 0$ and $v$ non-zero with $\theta(v) = \lambda.v$. Therefore the $G$-module map $\theta - \lambda.Id : V \to V$ has a non-zero kernel, and so its kernel must be the entirety of $V$. Therefore $\theta = \lambda.Id$. □

**Proposition 3.4.** *The regular module $k[G]$ is the free $G$-module on the set with one element* $1$.

*Proof.* Let $1 = \{\bullet\}$ be the singleton set – we have the map $e : 1 \to k[G] : \bullet \mapsto e$. Let $V$ be a $G$-module and $v : 1 \to V : \bullet \mapsto v$. We must show that there exists a unique $G$-module map $\theta : k[G] \to V$ such that the triangle

$$\begin{array}{ccc} k[G] & & \\ {\scriptstyle e}\uparrow & \searrow^{\theta} & \\ 1 & \xrightarrow[v]{} & V \end{array}$$

commutes. This triangle says that $\theta(e) = v$. If $\theta$ is to be a homomorphism we must have $\theta(g.e) = g.\theta(e) = v$ – thus we4 are forced to define $\theta(g) = e$ – since the elements of $G$ form a basis for the vector space $k[G]$ there is then a unique linear transformation $\theta : k[G] \to V$ satisfying $\theta(g) = g.v$ – it sends $\lambda_1 g_1 + \ldots + \ldots \lambda_n g_n$ to $\lambda_1(g_1.v) + \ldots + \lambda_n(g_n.v)$. It is clearly a $G$-module map. □

**Corollary 3.5.** *Let $G$ be a finite group and $k$ a field such that $char(k)$ does not divide $|G|$ – e.g. if $k$ is infinite. Each short exact sequence of $G$-modules splits.*

*Proof.* Consider

$$0 \longrightarrow U \xrightarrow{j} V \xrightarrow{k} W \longrightarrow 0$$

short exact. By Maschke's theorem $V = im(U) \oplus W^*$ for some $G$-module $W^*$. Then we have the projection

$$V = im(U) \oplus W^\star \to im(U) : v + w \mapsto v$$

and its composite with the isomorphism $j^{-1} : im(U) \cong U$ gives a splitting for $j$. Therefore the short exact sequence splits. □

**Corollary 3.6.** *Let $G$ be a finite group, such that $char(k)$ does not divide $|G|$. Let $k[G] = U_1 \oplus \ldots U_n$ have each $U_i$ irreducible. Then each irreducible $G$-module $U$ is isomorphic to one of the $U_i$.*

*Proof.* Let $v \neq 0 \in V$. By the universal property of $k[G]$ there exists a unique $G$-module map $\theta : k[G] \to V$ with $\theta(e) = v$. Since $V$ is irreducible, we must have that $im(\theta) = V$. Now we then have $ker(\theta) \leq k[G]$. Since $0 \to ker(\theta) \hookrightarrow k[G] \to im(\theta) \to 0$ is a short exact sequence, and each short exact sequence of $G$-modules splits we have $k[G] \cong ker(\theta) \oplus V$. In particular $V$ is then a direct summand of $k[G]$ and so isomorphic to a submodule of $k[G]$.

It thus suffices to prove the claim in the case that $U$ is a submodule of $k[G]$. If $U_i$ is trivial it is clear. Otherwise, let $p_i : k[G] \to U_i : u_1 + \ldots + u_n \mapsto u_i$ denote the projection on to each factor. Since $U_i$ is non-zero one of the composites $U \to k[G] \to U_i$ must be non zero, and so – by the first part of Schur's lemma – an isomorphism. □

**Corollary 3.7.** *Let $G$ be a finite group, such that $char(k)$ does not divide $|G|$. Then there are only finitely many irreducible $G$-modules over $k$, up to isomorphism.*

## 4. Finer results on decompositions when $k = \mathbb{C}$

**Terminology 4.1.** We call $V_1, \ldots, V_m$ a complete set of irreducible $G$-modules no two are isomorphic, and each irreducible $G$-module is isomorphic to one of them.

Our goal now is to take a closer look at the decomposition of a $G$-module into its irreducible constituents. The following is an immediate consequence of Schur's lemma.

**Definition 4.2.** Let $V$ and $W$ be $G$-modules. We write $Hom_{k[G]}(V, W) \subset Vect(V, W)$ for the vector subspace of $G$-module homomorphisms from $V$ to $W$. Warning – this is not a sub-$G$-module in general, though if $G$ is commutative, it is.

**Corollary 4.3.** *Let $V$ and $W$ be irreducible $G$-modules. Then*

$$dim(Hom_{\mathbb{C}[G]}(V, W)) = 1$$

*iff $V$ and $W$ are isomorphic and* 0 *otherwise.*

*Proof.* To say that $dim(Hom_{\mathbb{C}[G]}(V, W)) = 0$ is to say that the only homomorphism from $V$ to $W$ is the zero homomorphism. By Schur's lemma is equivalent saying that $V$ and $W$ are non-isomorphic.

If $dim(Hom_{\mathbb{C}[G]}(V, W)) = 1$ then there exists a non-zero homomorphism from $V$ to $W$ which is then an isomorphism by Schur's lemma. On the other hand if $\varphi : V \to W$ is an isomorphism of $G$-modules, then post-composition by $\varphi$ gives an isomorphism of vector spaces $Hom_{\mathbb{C}[G]}(V, V) \cong Hom_{\mathbb{C}[G]}(V, W))$. Now $dim(Hom_{\mathbb{C}[G]}(V, V)) = 1$ by Schur's lemma Part 2 – that is, each non-zero linear endomorphism is a scalar multiple of the identity – hence $dim(Hom_{\mathbb{C}[G]}(V, W)) = 1$ too. □

**Proposition 4.4.** *(1) For all $G$-modules we have*

$$Hom_{k[G]}(V_1 \oplus \ldots \oplus V_n, W) \cong Hom_{k[G]}(V_1, W) \oplus \ldots \oplus Hom_{k[G]}(V_n, W)$$

*and*

$$Hom_{k[G]}(V, W_1 \oplus \ldots \oplus W_n) \cong Hom_{k[G]}(V, W_1) \oplus \ldots \oplus Hom_{k[G]}(V, W_n)$$

.

*(2) We have an isomorphism of vector spaces $Hom_{k[G]}(k[G], V) \cong V$.*

*Proof.* The first part is standard and simply expresses the fact that direct sums of modules are *biproducts*. The universal property of $k[G]$ as the free $G$-module on 1 asserts that restriction along $e : 1 \to k[G]$ induces a bijection $Hom_{k[G]}(k[G], V) \to \mathbf{Set}(1, V) \cong V$. This composite bijection sends $\theta$ to the value $\theta(e)$ and is clearly a linear transformation. Therefore it is an isomorphism of vector spaces. □

**Theorem 4.5.** *Let $V$ be a finite dimensional $G$-module over $\mathbb{C}$.*

*(1) Then $V$ admits a decomposition $V = V_1 \oplus \ldots V_n$ as a direct sum of irreducible submodules.*
*(2) Each irreducible $G$-module $W$ appears in this decomposition, up to isomorphism, exactly $dim(Hom_{\mathbb{C}G}(V, W))$ times.*
*(3) In particular, let $U_1, \ldots, U_m$ be a complete set of irreducible $G$-modules. Then $V \cong U_1^{d_1} \oplus \ldots \oplus U_m^{d_m}$ where $d_i = dim(Hom_{\mathbb{C}G}(V, U_i))$.*

*Proof.* (1) This is just Theorem 3.1.

(2) We have

$$Hom_{\mathbb{C}[G]}(V, W) = Hom_{\mathbb{C}[G]}(V_1 \oplus \ldots V_n, W) \cong \\ Hom_{\mathbb{C}[G]}(V_1, W) \oplus \ldots \oplus Hom_{\mathbb{C}[G]}(V_n, W)$$

By Schur's lemma $dim(Hom_{\mathbb{C}[G]}(V_i, W)) = 1$ iff $V_i$ is isomorphic to $W$ and 0 otherwise. Hence, taking dimensions on either side of the above isomorphism gives $dim(Hom_{\mathbb{C}G}(V, W) = \Sigma_{i:V_i \cong W} 1$, which is the number of $i$ for which $V_i \cong W$, as claimed.

(3) Using the previous part, the direct sum $V = V_1 \oplus \ldots V_n$ is a direct sum

$$(V_{11} \oplus \ldots \oplus V_{1m_1}) \oplus (V_{21} \oplus \ldots \oplus V_{2m_2}) \oplus \ldots \oplus (V_{n1} \oplus \ldots \oplus V_{nm_n})$$

of irreducibles where we gather the isomorphic $G$-modules in the decomposition inside the same bracket and where $m_i = dim(Hom_{\mathbb{C}G}(V, V_{i1}))$. Since all of those within a single bracket are isomorphic to a common $U_i$, and none outside are, this gives $V \cong U_1^{d_1} \oplus \ldots \oplus U_m^{d_m}$.

□

Specialising the above to deal with the case of the regular $G$-module, we obtain a still finer result.

**Theorem 4.6.** *Let $\mathbb{C}[G] = V_1 \oplus \ldots V_n$ be a decomposition with each $V_i$ irreducible.*

*(1) Then each irreducible $G$-module $W$ appears in this decomposition, up to isomorphism, exactly $dim(W)$ times.*
*(2) In particular, let $U_1, \ldots, U_m$ be a complete set of irreducible $G$-modules. Then $\mathbb{C}[G] \cong U_1^{dim(U_1)} \oplus \ldots \oplus U_m^{dim(U_m)}$.*

*Proof.* This follows immediately from the above given that we have an isomorphism of vector spaces $Hom_{\mathbb{C}G}(\mathbb{C}G, W) \cong W$ for all $W$. □

The following result is very useful for calculating the irreducible representations of a given group.

**Theorem 4.7.** *Let $G$ be a finite group and $U_1, \ldots U_m$ a complete set of irreducible $G$-modules over $\mathbb{C}$. Then $|G| = \Sigma_{i=1,\ldots m} dim(V_i)^2$.*

*Proof.* This result follows from the second part of the above, on comparing the dimension of the left and right hand side. □

A simple consequence of the above results is the following.

**Proposition 4.8.** *A finite group is abelian if and only if all of its complex irreducible representations are* 1*-dimensional.*

*Proof.* Let $G$ be abelian and consider an irreducible $G$-module $V$. Let $g \in G$. Then $g.- : V \to V$ is a homomorphism of $G$-modules since $g.(h.v) = (g.h).v = (h.g).v = h.(g.v)$. Therefore, by Schur's lemma, there exists $\lambda$ such that $g.v = \lambda.v$ for all $v \in V$. But then each 1-dimensional subspace $\langle v \rangle \leq V$ is a $G$-submodule – hence, by irreducibility we have $\langle v \rangle = V$, as required.

Let $V$ be a 1-d representation. Since endomorphisms of $V$ are just multiplication by a scalar, their composition is commutative – hence given $g, h \in G$ we have $(gh).v = (hg).v$. Now let $\mathbb{C}[G] = V_1 \oplus \ldots \oplus V_n$ be a decomposition as a sum of 1-dimensional representations. (We know that $\mathbb{C}[G]$ is a direct sum of irreducibles). Then for $gh.(v_1 \oplus \ldots \oplus v_n) = (gh.v_1 \oplus \ldots \oplus gh.v_n) = (hg.v_1 \oplus \ldots \oplus hg.v_n) = hg.(v_1 \oplus \ldots \oplus v_n)$. Since $gh.- = hg.- : \mathbb{C}[G] \to \mathbb{C}[G]$ and $\mathbb{C}[G]$ is a faithful $G$-module we conclude that $gh = hg$. □

Let us calculate the 1-dimensional representations of a finite abelian group. Firstly, recall:

**Proposition 4.9.** *For a finite abelian group $G = C_{n_1} \times \ldots \times C_{n_m}$ it has $n_1 \times \ldots \times n_m$ non-isomorphic irreducible complex representations – all one dimensional. They are given by the assignments $(g_1^{i_1}, \ldots, g_m^{i_m}) \mapsto ({\lambda_1}^{i_1}, \ldots, {\lambda_m}^{i_m})$ where the $\lambda_i$ are $n_i$'th roots of unity.*

*Proof.* The first claim follows from the fact that all of its irreducible reps are 1-dimensional, and Theorem 4.7.

The above functions $G \to \mathbb{C}$ are clearly homomorphisms and indeed the only such – such each element of order $k$ must be sent to an $k$'th root of unity. They are also distinct – or, equally, non isomorphic. For isomorphism of 1-dimensional representations on the same space means equality $\lambda.(g._1 v) = g._2(\lambda.v) = \lambda(g._2 v)$ so $g._1 v = g._2 v$. Therefore they form a complete set of irreducible representations of $G$. □

**Example 4.10.** Representations of $D_8$. By the above theorem, the possible dimensions of the irreducible representations of any group of order 8 are $1, 1, 1, 1, 1, 1, 1, 1$ or $1, 1, 1, 1, 2$. In fact, we saw that $D_8$ has an irreducible $2 - d$ representation earlier. Hence it must have four irreducible 1-dimensional representations. Thus $D_8$ has 1 irreducible 2-d representation and 4 irreducible non-isomorphic 1-d representations.

What are the four 1-dimensional representations? To give a 1-dimensional representation is to give a homomorphism $D_8 \to \mathbb{C}$ – since $\mathbb{C}$ is abelian we obtain a unique factorisation

$$\begin{array}{ccc} D_8 & & \\ {\scriptstyle p}\downarrow & \searrow^{\rho} & \\ (D_8)_{ab} & \xrightarrow{\hat{\rho}} & \mathbb{C} \end{array}$$

through the abelianization of $D_8$. This gives a bijection, obtained by restricting along the quotient map $D_8 \to (D_8)_{ab}$, between the irreducible representations of the abelianization and the irreducible 1-dimensional representations of $D_8$. In the abelianization we force $ab = ba$. Since $b^{-1}ab = a^3$ this forces $a = a^3$ and so $a^2 = 1$. In this way, we see that $(D_8)_{ab} \cong C_2 \times C_2$ with quotient map $p : D_8 \to (D_8)_{ab} \cong C_2 \times C_2 : a \mapsto (1, 0), b \mapsto (0, 1)$ so that the irreducible representations of $D_8$ are obtained from the four irreducible representations of $C_2 \times C_2$ obtained by restricting along $p$ – namely $\rho(a) =^{+}_{-} 1$ and $\rho(b) =^{+}_{-} 1$.

## 5. Tensor products, internal homs and duals

5.1. **Tensor products.** Let $V$ and $W$ be vector spaces. The tensor product $V \otimes W$ can be described in various ways. I will describe a few and discuss a few standard facts about the tensor product, and their relationship with homs, and duals.

(1) Let $Bil(V, W; A)$ be the set of bilinear maps $F : V \times W \to A$ – recall that $F$ is bilinear if each $F(v, -) : V \to A$ and $F(-, w) : V \to A$ are linear maps. Then $V \otimes W$ classifies bilinear maps out of $V \times W$: more precisely, there exists a bilinear map $\theta : V \times W \to V \otimes W$ with the *universal property* that: given any bilinear map $F : V \times W \to A$ there exists a unique linear map $\tilde{F} : V \otimes W \to A$ such that

$$\begin{array}{ccc} V \times W & & \\ {\scriptstyle \theta}\downarrow & \searrow^{F} & \\ V \otimes W & \xrightarrow[\tilde{F}]{} & A \end{array}$$

This characterises the tensor product up to a unique isomorphism of vector spaces.

(2) Explicitly, $V \otimes W$, can be described as the quotient $F(V \times W)/\simeq$ of the free vector space on the underlying set of $V \times W$ quotiented by the relations:
- $\lambda v \otimes w = \lambda(v \otimes w) = v \otimes \lambda w$;
- $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$;
- $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$

wherein we write $v \otimes w$ for a basis element of $F(V \times W)$. In these terms we have $\theta(v, w) = v \otimes w$ and the above equations simply assert that $\theta$ is, indeed, bilinear.

(3) If $V$ has basis $v_i$ and $W$ basis $w_j$ then $V \otimes W$ has a basis consisting of the symbols $v_i \otimes w_j$ – in particular, if $V$ has dimension $n$ and $W$ has dimension $m$ then $V \otimes W$ has dimension $n \times m$.

(4) Given linear maps $T : V \to V$ and $S : W \to W$ represented by matrices $A$ and $B$ the corresponding matrix representing $T \otimes S$ is the *Kronecker product*

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{pmatrix}$$

In particular, it clearly follows that $Tr(A \otimes B) = Tr(A)Tr(B)$.

(5) The tensor is associative and symmetric up to *natural isomorphism*: we have $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ and $A \otimes B \cong B \otimes A$. The second isomorphism follows easily from the universal property. For the first isomorphism, one should show that both sides have the universal property of representing *trilinear* maps. A similar argument show that $k \otimes A \cong A \cong A \otimes k$ so that we have a kind of (up to isomorphism) commutative monoidal structure on **Vect** – this is called a *symmetric monoidal category* or *symmetric tensor category.*

We now explain how to lift the tensor product of $G$-modules to $G$-modules. Let $Bil_G(V, W; A)$ be the set of bilinear maps $F : V \times W \to A$ satisfying $g.F(v, w) = F(gv, gw)$ for all $v \in V$ and $w \in W$.

**Proposition 5.1.** *(1) The tensor product $V \otimes W$ becomes a $G$-module when we define $g(v \otimes w) = gv \otimes gw$.*
*(2) It has the universal property that* G-Mod$(V \otimes W, A) \cong Bil_G(V, W; A)$.

*Proof.* Consider the diagram

$$\begin{array}{ccc} V \times W & \xrightarrow{g_V \times g_W} & V \times W \\ \theta \big\downarrow & & \big\downarrow \theta \\ V \otimes W & \xrightarrow[g.-]{} & V \otimes W \end{array}$$

and observe that the composite $\theta(g_V \times g_W) : (v, w) \mapsto gv \otimes gw$ is bilinear. Therefore we obtain a unique linear map $g.- = g_V \otimes g_W : V \otimes W \to V \otimes W$ which is given by $g(v \otimes w) = gv \otimes gw$. That this action is compatible with the group structure of $G$ follows from the universal property of $V \otimes W$.

For the second part, we observe that a $G$-module map $\theta : V \otimes W \to A$ corresponds to a bilinear map $\varphi : V \times W \to A$, with the extra property that the diagram

$$\begin{array}{ccc} V \times W & \xrightarrow{g_V \times g_W} & V \times W \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ A & \xrightarrow[g.-]{} & A \end{array}$$

commutes – this says exactly that $\theta$ is a $G$-bilinear map. □

**Remark 5.2.** Using this universal property (as well as its ternary version) we can see that the isomorphisms $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ and $A \otimes B \cong B \otimes A$ of vector spaces lift to isomorphisms of $G$-modules. The isomorphisms $k \otimes A \cong A \cong A \otimes k$ also lift to isomorphisms of $G$-modules when $k$ is equipped with the *trivial $G$-module structure.*

5.2. **Homs and their relationship with tensor products.** Given vector spaces $V$ and $W$ we have the hom vector space $[V, W] = \mathbf{Vect}(V, W)$ – its structure is given by pointwise addition and scalar multiplication of linear maps. It is closely related to the tensor product of vector spaces by the fact that we have a bijection

$$\mathbf{Vect}(V \otimes W, A) \cong \mathbf{Vect}(V, [W, A])$$

natural in each variable. This follows from the fact that we have an isomorphism $Bil(V, W; A) \cong \mathbf{Vect}(V, [W, A])$ sending the bilinear

$$F : V \times W \to A$$

to the linear map

$$\tilde{F} : V \to \mathbf{Vect}(W, A) : v \mapsto (w \mapsto F(v, w)).$$

Using this, we can see that the composite isomorphism

$$\mathbf{Vect}(V \otimes W, A) \cong Bil(V, W; A) \cong \mathbf{Vect}(V, \mathbf{Vect}(W, A))$$

sends $F : V \otimes W \to A$ to the map

$$V \to \mathbf{Vect}(W, A) : v \mapsto (w \mapsto F(v \otimes w)).$$

**Remark 5.3.** In fact, the tensor product and hom determine one another up to unique isomorphism in this way – that is, the bijection says that we have adjoint functors $- \otimes W \dashv [W, -]$.

We have seen that if $V$ and $W$ are $G$-modules then $V \otimes W$ is naturally a $G$-module. We would now like to describe a $G$-module structure on $[V, W]$ so that the natural bijection

$$\mathbf{Vect}(V \otimes W, A) \cong \mathbf{Vect}(V, [W, A])$$

lifts to a natural bijection

$$\text{G-Mod}(V \otimes W, A) \cong \text{G-Mod}(V, [W, A]).$$

This means that we would like to equip $[W, A]$ with a $G$-module structure such the square of vector spaces on the left commutes

$$\begin{array}{ccc} V\otimes W & \xrightarrow{F} & A \\ {\scriptstyle g_V\otimes g_W}\downarrow & & \downarrow{\scriptstyle g_A} \\ V\otimes W & \xrightarrow[F]{} & A \end{array} \qquad\qquad \begin{array}{ccc} V & \xrightarrow{\tilde F} & [W,A] \\ {\scriptstyle g_V}\downarrow & & \downarrow{\scriptstyle g_{[W,A]}} \\ V & \xrightarrow[\tilde F]{} & [W,A] \end{array}$$

if and only if the square on the right above commutes. To see how we should do this, consider a diagram as on the left below.

$$\begin{array}{ccc} V\otimes W & \xrightarrow{F} & A \\ {\scriptstyle R\otimes S}\downarrow & & \downarrow{\scriptstyle T} \\ V'\otimes W' & \xrightarrow[F']{} & A' \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\tilde F} & [W,A] \\ & & \downarrow{\scriptstyle [W,T]} \\ {\scriptstyle R}\downarrow & & [W,A'] \\ & & \uparrow{\scriptstyle [S,A']} \\ V' & \xrightarrow[\tilde F']{} & [W',A'] \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\tilde F} & [W,A] \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle [S^{-1},T]} \\ V' & \xrightarrow[\tilde F']{} & [W',A'] \end{array}$$

This commutes if and only the diagram in the middle commutes – here $[W, T]$ means postcomposition by $T$, and $[S, A']$ means precomposition by $S$. In the case that $S$ is invertible, this can be rephrased as the commutativity of the square on the right. Finally this tells that we should define the hom $G$-module as follows.

**Proposition 5.4.** *Let $V$ and $W$ be $G$-modules. Then $[V, W]$ becomes a $G$-module, where an element $g$ acts as $[g_V^{-1}, g_W] : [V, W] \to [V, W]$. Equationally, this says $(gf)(v) = g.f(g^{-1}(v))$.*

In terms of commutative diagrams it says

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle g^{-1}.-}\uparrow & & \downarrow{\scriptstyle g.-} \\ V & \xrightarrow{g.f} & W \end{array}$$

*Proof.* Certainly each $g.f \in \mathbf{Vect}(V, W)$ as it is a composite of three linear maps. It is easy to see that $g.-$ is itself a linear map, and $((hg).f)(v) = hg.f.(g^{-1}h^{-1})(v) = h(gfg^{-1})h^{-1}(v) = (h.(g.f))(v)$. □

It then follows from the above discussion that we have a bijection

$$\text{G-Mod}(V \otimes W, A) \cong \text{G-Mod}(V, [W, A])$$

natural in each variable, as required.

**Remark 5.5.** Observe that the above construction of a hom certainly doesn't work if we are considering representations of a general algebra or monoid – in order to lift the hom from vector spaces, we very much used that the elements of a group are invertible. More generally, it can be done for an algebra which is Hopf – see the final section for an outline of the construction in that setting.

### 5.3. **Homs via tensors and duals.**

**Remark 5.6.** The *dual representation* $V^* = [V, k]$ is the special case of the above, on taking $W = k$ with the trivial $G$-module structure. In particular, the action on the dual space is defined by $(gf)(v) = f(g^{-1}v)$ or just $g.- = \mathbf{Vect}(g^{-1}, k) : \mathbf{Vect}(V, k) \to \mathbf{Vect}(V, k)$.

Recall that if $V$ is finite dimensional with basis $E = e_1, \ldots, e_n$ then $V^*$ has basis $E^* = (e_1)^*, \ldots, (e_n)^*$ where $(e_i)^* : V \to k$ is the map sending $e_i$ to 1 and all other basis vectors to 0.

For all $V, W$ the *evaluation map*

$$V \otimes [V, W] \to W : a \otimes f \mapsto fa$$

is clearly a $G$-module map. In particular, this gives us a map $ev : V \otimes V^* \to k$ sending $f \otimes v \mapsto f(v)$.

We would like to prove that for $V, W$ finite dimensional $G$-modules we have $V^\star \otimes W \cong [V, W]$.

The following proof is based on a nice argument of Mária Šimková in the exercises, and simpler than the one I originally gave in class. We have the composite $G$-module map

$$V^\star \otimes W \otimes V \xrightarrow{\cong} V^\star \otimes V \otimes W \xrightarrow{ev \otimes 1} k \otimes W \xrightarrow{\cong} W$$

which sends $f \otimes w \otimes v$ to $f(v)w$. Using the natural bijection G-Mod$(V^\star \otimes W \otimes V, W) \cong$ G-Mod$(V^\star \otimes W, [V, W])$ this corresponds to a $G$-module map $\varphi : V^\star \otimes W \to [V, W]$.

**Proposition 5.7.** *For $V, W$ finite dimensional $G$-modules the map $\varphi : V^\star \otimes W \to [V, W]$ is invertible.*

*Proof.* Let $(v_i)_{i \in I}$ and $(w_j)_{j \in J}$ form bases of $V$ and $W$ respectively. Then the elements $v_i^* \otimes w_j$ form a basis of $V^\star \otimes W$. On the other hand, the elements $\chi_j^i$ of $[V, W]$ defined by $\chi_j^i(v_i) = w_j$, and 0 otherwise, form a basis of $[V, W]$. It is easy to see that $\varphi(v_i^* \otimes w_j) = \chi_j^i$; thus $\varphi$ is a bijection on basis elements, and so an isomorphism of vector spaces. □

**Remark 5.8.** In future, it would be nice to consider the relationship with *duals* in a symmetric monoidal category.

## 6. Characters of groups

In the present section, we will assume that $G$ is a finite group, that $k = \mathbb{C}$ and that all $G$-modules are finite dimensional. Let $A$ be a $n \times n$-matrix. The trace $tr[A] = \Sigma_{i=1,\dots n} a_{ii}$ is the sum of the diagonal entries.

**Exercise 6.1.** Use Proposition 5.7 and the evaluation map $V^* \otimes V \to \mathbb{C}$ to give a matrix-free formulation of the trace.

**Proposition 6.2.** *Some standard properties of the trace are as follows.*

(1) $tr(A + B) = tr(B + A)$,
(2) $tr(AB) = tr(BA)$,
(3) $tr(B^{-1}AB) = tr(A)$,
(4) $tr(A \oplus B) = tr(A) + tr(B)$,
(5) $tr(A \otimes B) = tr(A) \times tr(B)$.

*Proof.* The proof of the first two is elementary – whilst the third follows from the second. The fourth and fifth hold since the direct sum and tensor product of matrices are of the form

$$A \oplus B = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right] \qquad A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{pmatrix}$$

respectively. □

**Definition 6.3.** Let $V$ be a $G$-module with basis $B$ of dimension $n$. Then each $g \in G$ gives rise to a matrix $[g]_B \in Gl_n(\mathbb{C})$. The character of $V$ is the function $\chi : G \to \mathbb{C} : g \mapsto tr[g]_B$.

The character of $V$ is independent of the choice of basis, since if $C$ is a second basis, then $[g]_C = T^{-1}[g]_B T$ for the base change matrix, and then by (3) above, the claim follows.

**Terminology 6.4.** Let $V$ be a $G$-module with character $\chi$.

(1) The *degree* of $\chi$ is the dimension of $V$ – this is equally the value $\chi(1)$ of the identity element.
(2) The character $\chi$ is said to be *irreducible* if $V$ is irreducible.

**Proposition 6.5.**
*Isomorphic $G$-modules have the same character.*

*Proof.* If $V$ and $W$ are isomorphic, with bases $B$ and $C$, then the matrix representations $[g]_B$ and $[g]_C$ are equivalent – that is, $T^{-1}[g]_B T = [g]_C$ for an invertible $T$. Hence $tr[g]_B = tr[g]_C$. □

We will prove a converse to the above: that is, if the characters of $V$ and $W$ are equal then $V$ and $W$ are isomorphic. The strategy is as follows.

(1) We introduce an inner product $\langle -,-\rangle$ on the complex vector space $[G,\mathbb{C}]$ of functions from $G$ to $\mathbb{C}$. This will have the property that $dim(Hom_{\mathbb{C}G}(V,W)) = \langle \chi_V, \chi_W\rangle$. A consequence of Schur's lemma is hen that distinct irreducible characters $\chi$ and $\varphi$ are *orthonormal*.
(2) Now by Theorem 4.5 the $G$-module $V$ is isomorphic to direct sum $V \cong \oplus_{i=1,\ldots,n} V_i^{d_i}$ for $V_i$ a complete set of irreducibles, and $d_i = dim(Hom_{\mathbb{C}G}(V,V_i))$. But since $dim(Hom_{\mathbb{C}G}(V,V_i)) = \langle \chi, \chi_i\rangle$ where the $\chi_i$ are the irreducible characters corresponding to the $V_i$ this tells us that $V$ can be recovered from its character $\chi$.

The approach we take heavily uses the results of the preceding section on tensor products, homs and duals of representations. Before that, lets derive a few simple properties of characters.

**Proposition 6.6.** *Let $G$ be a finite group and $V$ a finite dimensional $G$-module over $\mathbb{C}$. If $g$ is order $n$ then there is a basis $B$ of $V$ such that $[g]_B$ is a diagonal matrix, all of whose values are $n$-th roots of unity.*

*Proof.* Consider the cyclic group $\langle g\rangle \leq G$. Then viewing $V$ as a $\langle g\rangle$-module, and since this group is abelian, it splits as a direct sum $V = V_1 \oplus \ldots \oplus V_n$ of 1-d submodules. Taking a basis $v_1,\ldots,v_n$ we have $gv_i = \lambda_i v_i$ for $\lambda_i$ a $n$'th root of unity. This proves the claim. □

**Proposition 6.7.** *Basic facts about characters of finite groups.*
*(1) If $x$ and $y$ are conjugate elements of $G$ then $\chi(x) = \chi(y)$.*
*(2) If $x$ has order $n$ then $\chi(x)$ is a sum of $n$'th roots of unity.*
*(3) We have $\chi(x^{-1}) = \overline{\chi(x)}$.*

*Proof.* (2) For $V$ a $G$-module with basis $B$ and $y = g^{-1}xg$ then $\chi(y) = tr[y]_B = tr([g^{-1}xg]_B) = tr([g]^{-1}{}_B[x]_B[g]_B) = tr[x]_B = \chi[x]$, as required.
(3) By the corollary above, we know that there is a basis $B$ of $V$ with $[x]_B$ a diagonal matrix whose values $w_i$ are $n$'th roots of unity. Taking $tr[x]_B$ gives the result.
(4) $\chi(x)$ is a sum of $n$-th roots of unity $w = w_1 + \ldots + w_m$. Since $[x^{-1}]_B = [x]_B^{-1}$ we have $\chi(x^{-1}) = w_1^{-1} + \ldots + w_n^{-1} = \overline{w_1} + \ldots + \overline{w_n} = \overline{w_1 + \ldots w_n} = \overline{w}$, as required. □

**Proposition 6.8.** *Characters of sums, tensor products, duals and homs.*
*(1) $\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g)$,*
*(2) $\chi_{V\otimes W}(g) = \chi_V(g)\chi_W(g)$*
*(3) $\chi_{V^*}(g) = \overline{\chi_V(g)}$*
*(4) $\chi_{[V,W]}(g) = \overline{\chi_V(g)}.\chi_W(g)$*

*Proof.* The first and second follow from the corresponding facts for the trace function.

For (3) let $E = e_1,\ldots,e_n$ be a basis of $V$. In the dual basis $E^* = e_1^*,\ldots,e_n^*$ to that of $V$, a map $\mathbf{Vect}(f,k) : \mathbf{Vect}(V,k) \to \mathbf{Vect}(V,k)$ is represented by the transpose of the matrix $[f]_E$. Since the trace of the transpose is equal to that of the original matrix, we conclude that $\mathbf{Vect}(g^{-1}, k)$ has trace $\chi_V(g^{-1})$. By the basic properties of characters, this equals the complex conjugate $\overline{\chi_V(g)}$.

For (4) we use the isomorphism $[V, W] \cong V^* \otimes W$ of Proposition 5.7. Since the character is isomorphism invariant, we have $\chi_{[V,W]} = \chi_{V^*\otimes W}$. Then using the above equations we get $\chi_{V^*\otimes W}(g) = \chi_{V^*}(g)\chi_W(g) = \overline{\chi_V(g)}\chi_W(g)$. □

Our next goal is to describe $dim(Hom_{k[G]}(V, W))$ in terms of the characters of $[V, W]$. To start, observe that given any $G$-module $V$ we can form the vector subspace $V^G \leq V$ consisting of those $v$ for which $gv = v$ for all $g \in G$. This has a trivial $G$-module structure.

**Proposition 6.9.** *We have an equality* $\mathbf{Vect}(V, W)^G = Hom_{k[G]}(V, W)$ *of vector spaces.*

*Proof.* To say that $f : V \to W$ is fixed under the action of all $g \in G$ is to say that $f(v) = gf(g^{-1}v)$, or equally, that $gf(v) = f(g(v))$ all $v$. □

The following result is closely connected to Maschke's theorem.

**Lemma 6.10** (Projection formula)**.** *Let $V$ be a $G$-module.*

*(1) Then the linear map*

$$p : v \mapsto \frac{1}{|G|}\sum_{g\in G} g(v)$$

*is a projection with image $V^G$.*

*(2) In particular, we have*

$$dim(V^G) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)$$

.

*Proof.* To see that $h.p(v) = p(v)$ we use that $\sum_{g\in G} g = \sum_{g\in G} hg$ – namely,

$$h.p(v) = h.(\frac{1}{|G|}\sum_{g\in G} g(v)) = \frac{1}{|G|}\sum_{g\in G} hg(v) = \frac{1}{|G|}\sum_{g\in G} g(v) = p(v)$$

since $h(\sum_{g\in G} g) = (\sum_{g\in G} g)$. Thus $im(p) \leq V^G$. On the other hand if $h.v = v$ all $v$ we have $pv = \frac{1}{|G|}\sum_{g\in G}(v) = \frac{|G|}{|G|}v = v$. Thus $im(p) = V$. Finally $p^2v = ppv = pv$ since $pv \in V^G$ and $p$ is invariant on elements of $V^G$.
(2) Since we have a projection $p$, we can write $V = V^G \oplus W$ and $p = id_{V^G} \oplus 0_W$. Choosing any bases of $V^G$ and $W$ we get $tr(p) = dim(V^G)$ so that

$$dim(V^G) = tr(\frac{1}{|G|}\sum_{g\in G} g) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)$$

□

**Exercise 6.11.** Show how the procedure of averaging a projection used in Maschke's theorem can be obtained from the projection formula by considering the internal hom $G$-module $[V,V]$.

Combining the two preceding results we obtain

**Proposition 6.12.** *We have* $dim(Hom_{\mathbb{C}[G]}(V,W)) = \frac{1}{|G|}\sum_{g\in G}\chi_{[V,W]}(g)$.

Finally, putting it all together we have

**Theorem 6.13.** $dim(Hom_{\mathbb{C}[G]}(V,W)) = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}.\chi_W(g)$.

## 6.1. The standard inner product.

**Definition 6.14.** Let $f_1, f_2 : G \to \mathbb{C}$. We define

$$\langle f_1, f_2\rangle = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

It is a standard fact that this gives an inner product on $[G,\mathbb{C}]$ – in other words, that it is linear in the first variable, conjugate symmetric (meaning $\langle f,g\rangle = \overline{\langle g,f\rangle}$) and positive definite.

**Theorem 6.15.** *We have* $\langle\chi_V,\chi_W\rangle = dim(Hom_{\mathbb{C}[G]}(V,W))$. *(In particular, the value is always a natural number.)*

*Proof.* Actually, what Theorem 6.13 gives is the first equation of

$$dim(Hom_{\mathbb{C}[G]}(V,W)) = \langle\chi_W,\chi_V\rangle = \overline{\langle\chi_V,\chi_W\rangle} = \langle\chi_V,\chi_W\rangle.$$

We then use conjugate symmetry followed by the fact that the value is a natural number (being the dimension of a vector space). □

**Question 6.16.** In class Yuriy Dupyn asked me to what extent characters are a decategorification of representations – tensor product corresponds to multiplication, direct sum to addition etc. I'm especially curious about how internal homs induce inner products – can one show something like: the set of isomorphism classes of objects in a symmetric monoidal closed category with duals enriched in finite dimensional vector spaces admits an inner product – or something in that direction?

**Remark 6.17.** Note that since $\langle\chi_V,\chi_W\rangle$ is always a natural number, the inner product on $[G,\mathbb{C}]$ is in fact symmetric when restricted to characters – it follows that it is linear in both variables when restricted to characters.

The following important result is a routine consequence of what we have done.

Let $U_1,\ldots,U_m$ be a complete set of irreducible $G$-modules. Recall that $V \cong U_1^{d_1}\oplus\ldots\oplus U_m^{d_m}$ where $d_i = dim(Hom_{\mathbb{C}G}(V,U_i)) = \langle\chi_V,\chi_{U_i}\rangle$.

**Theorem 6.18.** *Two $G$-modules $V$ and $W$ are isomorphic if and only if they have the same character.*

*Proof.* We have proven one direction already. For the other, we have $V \cong U_1^{d_1} \oplus \ldots \oplus U_m^{d_m}$ where $d_i = \langle \chi_V, \chi_{U_i} \rangle = \langle \chi_W, \chi_{U_i} \rangle$. Hence we also have $W \cong U_1^{d_1} \oplus \ldots \oplus U_m^{d_m}$, as required. □

**Theorem 6.19.** *Irreducible characters are orthonormal – that is, if $V$ and $W$ are irreducible $G$-modules then $\langle \chi_V, \chi_W \rangle = 1$ iff $\chi_V = \chi_W$ and 0 otherwise.*

*Proof.* We have $\langle \chi_V, \chi_W \rangle = dim(Hom_{\mathbb{C}G}(V, W)$. By Schur's lemma, the rhs is 1 just when $V \cong W$ (equally $\chi_V = \chi_W$ by the preceding) and 0 otherwise. □

In fact, we can strengthen the above result to obtain a very useful tool for checking whether a given representation is irreducible!

**Theorem 6.20.** *A $G$-module $V$ is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.*

*Proof.* One direction holds by the preceding result – we must show that if $\langle \chi_V, \chi_V \rangle = 1$ then $\chi_V$ is irreducible.

Write $V \cong U_1^{d_1} \oplus \ldots \oplus U_m^{d_m}$. Then $\chi_V = d_1\chi_{U_1} + \ldots d_m\chi_{U_m}$ since the character of a direct sum is the sum of the characters. Using orthogonality of irreducible characters and linearity in each variable, we then get $\langle \chi_V, \chi_V \rangle = (d_1)^2 + \ldots + (d_n)^2$. Being a sum of natural numbers, this can equal 1 if and only if one of the $d_i$ equals 1, and all of the others are 0, as required. □

**Example 6.21.** Consider $D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. and its 2-dimensional representation by rotation and reflection matrices.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, A^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, AB = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, A^2B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, A^3B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

so its character is given by

| $g$ | 1 | $a$ | $a^2$ | $a^3$ | $b$ | $ab$ | $a^2b$ | $a^3b$ |
|---|---|---|---|---|---|---|---|---|
| $\chi$ | 2 | 0 | $-2$ | 0 | 0 | 0 | 0 | 0 |

In particular, we see that $\langle \chi, \chi \rangle = (2^2 + (-2)^2)/8 = 8/8 = 1$. This gives another proof that the 2-dimensional representation is irreducible.

### 6.2. **The number of irreducible characters.**

**Definition 6.22.** A *class function* $f : G \to \mathbb{C}$ is a function which is invariant on conjugacy classes.

Each character is a class function. Let $C(G) \leq [G, \mathbb{C}]$ be the vector subspace consisting of the class functions – we will show that the irreducible characters form a basis for $C(G)$.

**Lemma 6.23.** *The dimension of $C(G)$ is equal to the number of conjugacy classes of $G$.*

*Proof.* Consider the surjective function $p : G \to Conj(G) : g \mapsto [g]$ sending an element of $g$ to its conjugacy class. Since a function $G \to \mathbb{C}$ is a class function if and only if it factors through $p$, restriction along $p$ induces a bijection $p^* : [Conj(G), \mathbb{C}] \cong C(G)$. Clearly this respects the pointwise vector space structure and so is an isomorphism of vector spaces. The left hand side has dimension $Conj(G)$ – the set of conjugacy classes, and so the right does too. (Indeed transporting the canonical basis on $[Conj(G), \mathbb{C}]$ along $p^*$ we see that the functions $G \to C$ sending each member of a given conjugacy class to 1 and all other elements to 0 form a basis for $C(G)$.) □

We wish to prove that the number of irreducible characters – equally the number of isomorphism classes of irreducible $G$-modules – equals the number of conjugacy classes. Firstly, we need to consider the *centre of the group algebra.*

Recall that $\mathbb{C}[G]$ is an $\mathbb{C}$-algebra with multiplication $(\Sigma_j \lambda_i g_i)(\Sigma_j \theta_j g_j) = \Sigma_{i,j} \lambda_i \theta_j g_i g_j$. We observed earlier that a $G$-module $V$ is equivalently a $\mathbb{C}[G]$-module – in particular each $z = \lambda_1 g_1 + \ldots + \lambda_n g_n \in \mathbb{C}[G]$ induces a linear transformation $z.- = \lambda_1 g_1.- + \ldots + \lambda_n g_n.- : V \to V$ by pointwise operations.

The centre

$$Z(\mathbb{C}[G]) = \{z \in \mathbb{C}[G] : zw = wz \ \ \forall w \in \mathbb{C}[G]\}$$

is a vector subspace of $\mathbb{C}[G]$, though not necessarily a $G$-module. Clearly $z$ lies in the centre just when $zg = gz$ for each $g \in G$ – in particular, each element of the centre $Z(G)$ of the group $G$ belongs to $Z(\mathbb{C}[G])$ but others do too.

Note that if $g \in Z(G)$ then $g.- : V \to V$ is a $G$-module homomorphism since $g.(h.v) = (gh).v = (hg).v = h(gv)$. Arguing in the same way, we see more generally that if $z \in Z(\mathbb{C}[G])$ then $z.- : V \to V$ is a $G$-module homomorphism.

**Lemma 6.24.** *The centre of the group algebra $Z(\mathbb{C}G)$ has dimension equal to the number of conjugacy classes of $G$.*

*Proof.* Let $C_1, \ldots, C_n$ be the conjugacy classes of $G$. We define the *class sum* $\overline{C_i} = \Sigma_{g \in C_i} g$ and will show that the class sums form a basis for $Z(\mathbb{C}G)$, thus proving the claim.

Firstly, we show that $\overline{C_i} \in Z(\mathbb{C}G)$. In order to show this, it suffices to prove that $h.\overline{C_i} = \overline{C_i}.h$ for each $h \in G$ since these are the basis elements of $\mathbb{C}G$. Equivalently, $h\overline{C_i}h^{-1} = \overline{C_i}$ but since $h^{-1}. - .h : C_i \to C_i$ permutes the elements of the conjugacy class, the class sum is indeed invariant.

Next, we show that the $\overline{C_i}$ are linearly independent. For suppose that $\Sigma_i \lambda_i \overline{C_i} = 0$. This is a sum of the basis elements $g \in G$, and since the conjugacy classes are disjoint no two basis elements are repeated – it follows that each $\lambda_i = 0$.

Finally, we must show they are spanning. Let $r = \Sigma_g \lambda_g g$ lie in the centre. Then for all group elements $h$ we have $r = hrh^-1$, which says that

$$\Sigma_g \lambda_g g = \Sigma_g \lambda_g hgh^{-1}$$

– in particular we must have $\lambda_g = \lambda_{hgh^{-1}}$. In other words, the function $\lambda : G \to \mathbb{C} : g \mapsto \lambda_g$ is invariant on conjugacy classes (a class function). Therefore $r = \Sigma_g \lambda_g g = \Sigma_i \lambda_{g_i} C_i$ where $g_i$ is any member of the conjugacy class $C_i$. □

**Theorem 6.25.** *Let $V_1, \ldots, V_n$ be a complete set of irreducible $G$-modules, and $\chi_1, \ldots, \chi_n$ the associated characters form a basis of $C(G)$ – in particular the number of irreducible characters equals the number of conjugacy classes of $G$.*

*Proof.* Since the $\chi_i$ are orthogonal, they are linearly independent in $C(G)$. Hence $n \leq l$ the number of conjugacy classes, since $l$ is the dimension of $C(G)$. We must show that $n = l$. By the preceding lemma $l$ is equally the dimension of the centre of the group algebra $Z(\mathbb{C}G)$ – it suffices to prove that $dim(Z(\mathbb{C}G)) \leq n$.

Let $\mathbb{C}G = W_1 \oplus \ldots \oplus W_n$ where $W_i$ is isomorphic to a direct sum of copies of $V_i$. Consider the identity group element $e \in \mathbb{C}[G]$ – then we can write $e = f_1 + \ldots f_n$ where $f_i \in W_i$. Let $z \in Z(\mathbb{C}G)$ and consider

$$z.- : V \to V : v \mapsto z.v$$

for any $G$-module $V$. This is a $G$-module morphism since $z.gv = (z.g)v = (g.z)v = g(zv)$. In particular, by Schur's lemma we have $z.- = \lambda_i.- : V_i \to V_i$ for some $\lambda_i \in \mathbb{C}$. Furthermore we then have $z.- = \lambda_i : - : \oplus V_i \to \oplus V_i$ for any such direct sum, and since we have $W_i \cong \oplus V_i$ as $G$-modules, we also have that $z.- = \lambda_i.- : W_i \to W_i$.

Then $z = z.e = z.(f_1 + \ldots + f_n) = \lambda_1 f_1 + \ldots + \lambda_n f_n$ so that $dim(Z(\mathbb{C}G)) \leq n$, as required. □

## 6.3. **Character tables.**

**Definition 6.26.** Let $G$ be a group. The character table of $G$ is a square array with columns indexed by the conjugacy classes of $G$ and rows by the irreducible characters, as below.

$$\left[\begin{array}{c|c|c|c} & \dots & K & \dots \\ \hline \dots & & \vdots & \\ \chi & \dots & \chi_K & \\ \vdots & & & \end{array}\right]$$

The values are in the first column give the dimensions of the representations.

**Remark 6.27.** The values of the first column give the dimensions of the irreducible characters.

**Example 6.28** (Character table of $D_8$)**.** Consider $D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. Its conjugacy classes are $\{1\}, \{a, a^3\}, \{a^2\}, \{b, a^2b\}, \{ab, a^2b\}$. Its character table is

| $g$ | 1 | $a$ | $a^2$ | $b$ | $ab$ |
|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | 1 | $-1$ | $-1$ |
| $\chi_3$ | 1 | $-1$ | 1 | 1 | $-1$ |
| $\chi_4$ | 1 | $-1$ | 1 | $-1$ | 1 |
| $\chi_5$ | 2 | 0 | $-2$ | 0 | 0 |

There is one 2-dimensional representation and four 1-d representations. Note that the character of a 1-d representation is just the original representation – in particular it is multiplicative. We calculated these in Example 4.10.

What can we say about the character table? It is certainly a square matrix of dimension $Conj(G)$ the set of conjugacy classes of $G$. Recall that the irreducible characters form a basis of $C(G)$, the space of class functions of $G$. Using the isomorphism $C(G) \cong \mathbf{Vect}(Conj(G), \mathbb{C})$ we see that the restricted irreducible characters form a basis of $\mathbf{Vect}(Conj(G), \mathbb{C})$. Hence the matrix with these as its rows is invertible.

6.4. **Row and column orthogonality relations.** For irreducible characters $\chi_i, \chi_j$ we have, by their orthonormality, the equation

$$\delta_{i,j} = \frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_j(g)}.$$

Let $Cl(g)$ be the conjugacy class of $g \in G$, and $g_1, \dots, g_n$ a set of representatives of the conjugacy classes. Then since characters are invariant on conjugacy classes, the above equation becomes

$$\delta_{i,j} = \frac{1}{|G|}\sum_{g_k\in G}|Cl(g_k)|\chi_i(g_k)\overline{\chi_j(g_k)}.$$

These are called the *row orthogonality relations*, since they express the relation between different rows of the character table. Very useful are also the *column orthogonality relations*, which are stated as the second part of the following summary result.

**Theorem 6.29.** *Consider representatives $g_1, \ldots, g_k$ of the conjugacy classes of a finite group $G$. Then*

- *[Row orthogonality relations]*

$$\delta_{i,j} = \frac{1}{|G|} \sum_{g_k \in G} |Cl(g_k)| \chi_i(g_k) \overline{\chi_j(g_k)}.$$

- *[Column orthogonality relations]*

$$\frac{|G|}{|Cl(g_i)|} \delta_{i,j} = \sum_{k=1,\ldots,n} \chi_k(g_i) \overline{\chi_k(g_j)}.$$

*Proof.* We only need to prove the second case. Let

$$B_{i,j} = \sqrt{\frac{|Cl(g_j)|}{|G|}} \chi_i(g_j).$$

Then

$$(B.\overline{B}^t)_{ij} = \sum_{k=1,\ldots,n} \frac{|Cl(g_j)|}{|G|} \chi_i(g_k) \overline{\chi_j(g_k)} = \delta_{ij}$$

by row orthogonality, so $B.\overline{B}^t = I$. By the multiplicativity of the determinant, this implies that $B$ is invertible, so its inverse must be $\overline{B}^t$. Hence $\overline{B}^t.B = I$ too. This says

$$\delta_{ij} = (\overline{B}^t.B)_{ij} = \sum_{k=1,\ldots,n} \sqrt{\frac{|Cl(g_i)|}{|G|}} \sqrt{\frac{|Cl(g_j)|}{|G|}} \overline{\chi_k(g_i)} \chi_k(g_j)$$

which is equally

$$\frac{|G|}{|Cl(g_i)|} \delta_{ij} = \sum_{k=1,\ldots,n} \overline{\chi_k(g_i)} \chi_k(g_j) = \sum_{k=1,\ldots,n} \chi_k(g_i) \overline{\chi_k(g_j)}$$

using that both sides are real numbers in the last case. □

These are very useful for completing character tables given partial information. For instance consider again the character table for $D_8$ but with the final row and columns missing. Recall the conjugacy classes are

$\{1\}, \{a, a^3\}, \{a^2\}, \{b, a^2b\}, \{ab, a^2b\}$.

| $g$ | 1 | $a$ | $a^2$ | $b$ | $ab$ |
|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | ? |
| $\chi_2$ | 1 | 1 | 1 | $-1$ | ? |
| $\chi_3$ | 1 | $-1$ | 1 | 1 | ? |
| $\chi_4$ | 1 | $-1$ | 1 | $-1$ | ? |
| $\chi_5$ | ? | ? | ? | ? | ? |

| $g$ | 1 | $a$ | $a^2$ | $b$ | $ab$ |
|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | 1 | $-1$ | ? |
| $\chi_3$ | 1 | $-1$ | 1 | 1 | ? |
| $\chi_4$ | 1 | $-1$ | 1 | $-1$ | ? |
| $\chi_5$ | ? | ? | ? | ? | ? |

Firstly, the top row is the trivial character so we immediately fill in the top row above. Now using row orthogonality, we have

$$0 = \langle \chi_1, \chi_2 \rangle = 1/8(1.1 + 2(1.1) + 1(1.1) + 2(1. - 1) + 2(1.\chi_2(ab))$$

so we get $0 = 1/8(2 + 2(\chi_2(ab))$ so $\chi_2(ab) = -1$.

Continuing in this way, we fill in as far as on the left below

| $g$ | 1 | $a$ | $a^2$ | $b$ | $ab$ |
|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | 1 | $-1$ | $-1$ |
| $\chi_3$ | 1 | $-1$ | 1 | 1 | $-1$ |
| $\chi_4$ | 1 | $-1$ | 1 | $-1$ | 1 |
| $\chi_5$ | ? | ? | ? | ? | ? |

| $g$ | 1 | $a$ | $a^2$ | $b$ | $ab$ |
|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | 1 | $-1$ | $-1$ |
| $\chi_3$ | 1 | $-1$ | 1 | 1 | $-1$ |
| $\chi_4$ | 1 | $-1$ | 1 | $-1$ | 1 |
| $\chi_5$ | 2 | ? | ? | ? | ? |

The we use the fact that $8 = 1^2 + 1^2 + 1^2 + 1^2 + \chi_5(1)$ to deduce that $\chi_5(1) = 2$ – that is, the final irreducible representation has dimension 2. Then we can calculate the last row using column orthogonality. For instance, we have

$$0 = 1.1 + 1.1 + 1. - 1 + 1. - 1 + 2\chi_5(a) = 0 + 2\chi_5(a)$$

so $\chi_5(a) = 0$. And now we can fill in the rest.

Another useful trick for filling in the character table is the following one.

**Proposition 6.30.** *If $\chi$ is an irreducible character and $\theta$ a 1-dimensional character, then $\chi.\theta$ is also an irreducible character.*

*Proof.* By Theorem 6.20, we must show that $\langle \chi.\theta, \chi.\theta \rangle = 1$. Now this equals

$$\frac{1}{|G|} \sum_{g \in G} \chi(g)\theta(g)\overline{\chi(g)\theta(g)} = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)}\theta(g)\overline{\theta(g)} = \sum_{g \in G} \chi(g)\overline{\chi(g)} = 1$$

where the second equation uses since $\theta$ is 1-dimensional that $\theta(g)$ is a root of unity, and that if $x$ is a root of unity then $\overline{x} = x^{-1}$. □

**Example 6.31.** The *regular character* $\chi_{reg}$ is the character of the regular $G$-module $\mathbb{C}[G]$. What are its values? Consider $g.- : \mathbb{C}[G] \to \mathbb{C}[G]$ and its matrix representation $[g]$ with respect to the standard basis $\{g_1, \ldots, g_n\}$. Then $[g]$ is a permutation matrix – we have $[g]_{ij} = 1$ if $g.g_j = g_i$ and 0 otherwise. In particular, $[g]_{ii} = 0$ unless $[g] = e$, in which case $[g]_{ii} = 1$. Therefore, we see that $\chi_{reg}(e) = |G|$ whilst $\chi_{reg}(g) = 0$ otherwise.

## 7. Kan extensions, restricted and induced representations

Let $H \leq G$ be a subgroup. It is clear that if $V$ is a $G$-module then it becomes a $H$-module by restricting the action to elements of $H$ – one writes $V \downarrow H$ for the restricted module. Furthermore if $f : V \to W$ is a $G$-module map then it can be viewed as a $H$-module map $f : V \downarrow H \to W \downarrow H$ between the restricted $H$-modules – indeed we have a functor $Res$ : G-Mod $\to$ H-Mod.

All of this is obvious. What about the opposite direction? That is, from a $H$-module can we construct a $G$-module and what is the best way to do so? Category theory suggests the answer – look for a left adjoint to this functor. Its value at a $H$-module $V$ is called the *induced representation* $V \uparrow G$ and is characterised by a bijection

$$\text{G-Mod}(V \uparrow G, W) \cong \text{H-Mod}(V, W \downarrow H)$$

natural in $W$. The above formula, or actually its formulation for characters, is often called *Frobenius reciprocity.*

For a group $G$ let $G_\bullet$ be the corresponding category with one object $\bullet$ and $G_\bullet(\bullet, \bullet) = G$. The group multiplication defines composition in $G_\bullet$. As noted during the first week, a $G$-module is a functor $G_\bullet \to \mathbf{Vect}$. And indeed we can identify G-Mod with the category of functors and natural transformations G-Mod $= [G_\bullet, \mathbf{Vect}]$. The subgroup inclusion $H \leq G$ defines a functor $J : H_\bullet \to G_\bullet$ so that we obtain a restriction functor

$$Res = [J, 1] : [G_\bullet, \mathbf{Vect}] \to [H_\bullet, \mathbf{Vect}]$$

which we identify with $Res$ : G-Mod $\to$ H-Mod.

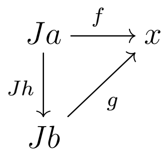
Left adjoints to such restriction functors are called left Kan extensions. More generally, given a functor $J : A \to B$ and a category $\mathcal{C}$ we obtain a functor $Res_J = [J, 1] : [B, \mathcal{C}] \to [A, \mathcal{C}]$ and can ask when it has a left adjoint. In other words, for $F : A \to \mathcal{C}$ we should find $Lan_J(F) : B \to \mathcal{C}$ and a natural transformation $\eta : F \to Lan_J(F) \circ J$ such that the induced function

$$[B, \mathcal{C}](L_J(F), G) \to [A, \mathcal{C}](F, GJ)$$

is invertible.

Diagramatically, this says To add.

The functor $Lan_J(F) : B \to \mathcal{C}$ is called the *left Kan extension* of $F$ along $J$. In fact there is a simple categorical criterion for when such left Kan extensions exist. Firstly, recall that given $b \in B$ and $J : A \to B$ we can form the *comma category* $J/x$ whose objects are pairs $(a \in A, f : Ja \to x)$. A morphism $h : (a, f) \to (b, g)$ is a morphism $h : a \to b \in A$ such that the triangle




There is a forgetful functor $P_x : J/x \to A : (a, f) \mapsto a$.

**Theorem 7.1.** *Let $J : A \to B$ be a functor from a small category to a locally small one, and consider $F : A \to C$. If for each $b \in B$ the colimit of*

$$J/x \xrightarrow{P_x} A \xrightarrow{F} B \qquad (a, f) \mapsto a \mapsto Fa$$

*exists, then $lan_J(F)$ exists, and has value at $x$ given by the above colimit $col(F \circ P_x)$. In particular, if $\mathcal{C}$ is cocomplete then the left Kan extension always exists.*

*Proof.* Add proof. □

**Theorem 7.2.**
- *The induced representation $V \uparrow G$ exists.*
- *If $G$ and $V$ are finite dimensional then $V \uparrow G$ can be described as follows. Let $G = t_1H \cup \ldots \cup t_nH$ decompose $G$ into cosets of $H$. Write $t_iV \cong V$ for an isomorphic copy of $V$ whose elements we write as $t_iv$ where $v \in V$. Then*

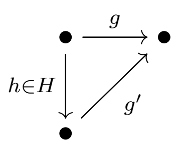
$$V \uparrow G = t_1V \oplus \ldots \oplus t_nV.$$

*Given $g \in G$ we should describe its action $g(\Sigma_i t_i v)$. We have $gt_i = t_{\sigma(i)}h_i$ for a unique transversal $t_{\sigma(i)}$ and $h_i \in H$. Then*

$$g(\Sigma_i t_i v) = \Sigma_i t_\sigma(i) h_i v.$$

*Proof.* The first part holds by the preceding theorem, on observing that the category of vector spaces has all small colimits – these can be constructed using (infinite) direct sums and cokernels.

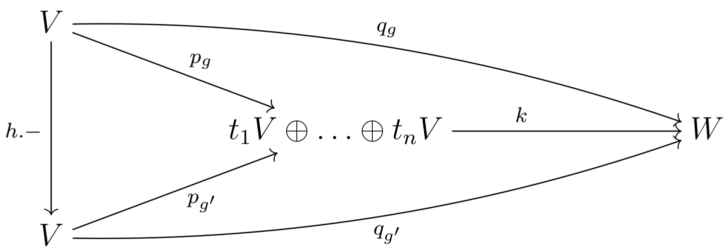
For the second part, consider $J : H_\bullet \to G_\bullet$. Now $G_\bullet$ has a single object $\bullet$. What does the slice category $J/\bullet$ look like? Its objects are morphisms $g : \bullet \to \bullet \in G_\bullet$ – that is elements of $G$. Morphisms $h : g \to g'$ are commutative triangles




where $h \in H$ – that is, $g'h = g$. Consider the functor

$$J \xrightarrow{P_\bullet} H_\bullet \xrightarrow{V} \mathbf{Vect}$$

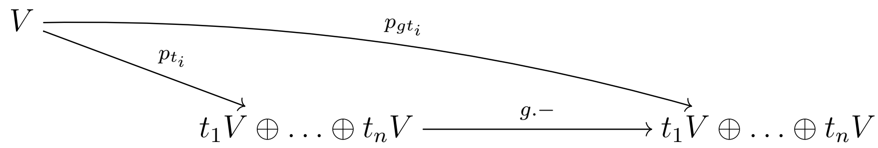
We must show, firstly, that the colimit is $t_1V \oplus \ldots \oplus t_nV$. For each $g \in G$ we must give a cocone map $p_g : V \to t_1V \oplus \ldots \oplus t_nV$. To define this, we use that $g = t_i.h_i$ for unique $t_i$ and $h \in H$ and then define $p_g(v) = t_i h.v$ – note that this is a linear map since $h.- : V \to V$ is. Next, we check that we have a cocone – namely that if $g'h = g$ as above, then the triangle left below commutes.




For this, let $v \in V$ and consider $p_{g'}(h.v)$. Now if $g = t_i.h_i$ then since $g = g'h$ we have $g' = gh^{-1} = t_i(h_i h^{-1})$. Therefore $p_{g'}(h.v) = t_i(h_i h^{-1})h.v = t_i h_i.v = p_g v$ as required.

Finally, suppose we have a second cocone $q$ to $W$ as above – we must show that there exists a unique map of cocones making the diagram commute. Observe that $p_{t_i}(v) = t_i v$ – thus since we must have $k \circ p_{t_i} = q_{t_i}$, and $k$ must be linear, we must set $k(\Sigma_i t_i v) = \Sigma_i q_{t_i} v$. Thus the definition of $k$ is forced on us – we must show that $k \circ p_g = q_g$ for all $g$. For $g = t_i h_i$ we have $kp_g v = k(t_i h_i.v) = q_{t_i}(h_i.v) = q_g v$ where the last equation holds by virtue of that $q$ is a cocone. This proves that it is the colimit as claimed.

By construction of the left Kan extension, the action satisfies




which says that $g(t_i v) = t_{\sigma(i)} h_i v$ where $gt_i = t_{\sigma(i)} h_i$, as claimed. □

**Remark 7.3.** The universal map of $H$-modules $\eta_V : V \to (V \uparrow G) \downarrow H$ is the map $p_e : V \to t_1 V \oplus \ldots \oplus t_n V$ where $e$ is the identity element of $G$. This is never invertible unless $H = G$ – the rhs has dimension $dim(V).|G|/|H|$. In particular, restriction along $\eta_V$ induces a bijection G-Mod$(V \uparrow H, W) \cong$ H-Mod$(V, W \downarrow H)$ – this is the established adjointness. If we view both left and right hand sides as vector spaces, with the pointwise structure, then since $\eta_V$ is linear, the bijection is in fact an isomorphism of vector spaces G-Mod$(V \uparrow H, W) \cong$ H-Mod$(V, W \downarrow H)$

**Theorem 7.4** (Frobenius reciprocity)**.** *For $H \leq G$ finite, $V$ a $H$-module, and $W$ a $G$-module over $\mathbb{C}$ we have*

$$\langle \chi_{V\uparrow G}, \chi_W \rangle = \langle \chi_V, \chi_{W\downarrow H} \rangle.$$

*Proof.* Recall that $\langle A, B\rangle = dim(\text{G-Mod}(A, B))$. Thus what the above says is that $dim(\text{G-Mod}(V \uparrow G, W)) = dim(\text{H-Mod}(\mathbb{C}H(V, W \downarrow H))$, which follows immediately from the remark above. □

**Example 7.5.** Consider the trivial $H$-module $\mathbb{C}$, where $H \leq G$. What is $\mathbb{C} \uparrow G$? According to the above formula it is $t_1K \oplus \ldots t_nK$ with action given by $gt_i = t_\sigma(i)$ where $gt_i = t_{\sigma(i)}h$ some $h$. Another way of looking at it is therefore that it is $\mathbb{C}[t_1, \ldots, t_n]$ which is the permutation representation corresponding to the left coset action of $G$ on $\{t_1, \ldots, t_n\}$.

Having already established Frobenius reciprocity, we can prove the following formula for the character of the induced representation. Let $V$ be a $H$-module with character $\varphi : H \to \mathbb{C}$. Let us write $\varphi \uparrow G$ for the character of $V \uparrow G$. To express this, we firstly extend the definition of $\varphi$ by defining a function

$$\dot{\varphi} : G \to \mathbb{C} : g \in H \mapsto \varphi(g), g \notin H \mapsto 0$$

**Proposition 7.6.** *The induced character satisfies the formula*

$$\varphi \uparrow G(g) = \frac{1}{|H|} \sum_{y \in G} \dot{\varphi}(y^{-1}gy)\ .$$

*Proof.* It is easy to see that the function $f : G \to \mathbb{C}$ on the right is a class function. Since the irreducible characters $\chi_1, \ldots, \chi_n$ form a basis of the class functions it suffices to show that the two class functions above have the same coefficients when written as linear combinations of the $\chi_i$ – equivalently that

$$\langle \varphi \uparrow G, \chi\rangle = \langle f, \chi\rangle$$

for each irreducible character $\chi$. Now by Frobenius reciprocity, the left hand side is equally $\langle \varphi, \chi \downarrow H\rangle$, which equals

$$\frac{1}{|H|} \sum_{x \in H} \varphi(x)\overline{\chi(x)} \quad .$$

The right hand side is

$$\frac{1}{|G|} \sum_{g \in G} f(g)\overline{\chi(g)} = \frac{1}{|G|}\frac{1}{|H|} \sum_{g \in G}\sum_{y \in G} \dot{\varphi}(y^{-1}gy)\overline{\chi(g)} \quad .$$

Let $x = y^{-1}gy$. Then the above equals

$$\frac{1}{|G|}\frac{1}{|H|} \sum_{x \in G}\sum_{y \in G} \dot{\varphi}(x)\overline{\chi(yxy^{-1})} = \frac{1}{|G|}\frac{1}{|H|} \sum_{x \in H} |G|\varphi(x)\overline{\chi(x)} = \frac{1}{|H|} \sum_{x \in H} \varphi(x)\overline{\chi(x)} \quad .$$

where in the first equality we have used that $\dot{\varphi}$ is zero if $x \notin H$ and that $\chi(yxy^{-1}) = \chi(x)$. □

## 8. Representation theory of the symmetric group

The number of irreducible representations of the symmetric group equals, by Theorem 6.25, the number of conjugacy classes. As is well known, the conjugacy classes of $S_n$ are parametrised by the cycle type of its elements.

We can think of cycle types as being specified by *partitions* of $n$, which are sequences $\lambda = (\lambda_1, \dots, \lambda_l)$ with $\lambda_i \geq \lambda_{i+1}$ and adding to $n$. One writes $\lambda \vdash n$ to indicated that $\lambda$ is a partition of $n$. For instance, the cycle type of the permutation (123)(45)(678) corresponds to the partition $(3, 3, 2)$. The partition can also be represented by an array

| | | |
|---|---|---|
| | | |
| | | |
| | | |

whose rows are of length the numbers in the partition. This array is called the shape of the partition. A Young tableaux of shape $\lambda \vdash n$ is an array $t$ of the given shape whose entries are bijectively filled by the numbers $1, \dots, n$. For instance, one Young tableau is

| | | |
|---|---|---|
| 2 | 3 | 1 |
| 4 | 7 | 8 |
| 5 | 6 | |

Evidently if $\lambda \dashv n$ there is a bijection between $\lambda$-tableaux and elements of $S_n$ – under this correspondence the above tableau corresponds to the permutation

$$1, 2, 3, 4, 5, 6, 7, 8 \mapsto 2, 3, 1, 4, 7, 8, 5, 6$$

where we read across the rows first. In particular, there are $n!$ $\lambda$-tableaux.

Furthermore, $S_n$ acts on the set of $\lambda$-tableau in the obvious way – that is, $(\pi t)_{i,j} = \pi(t_{i,j})$. In other words, $\pi$ acts on the entries of the tableaux – it doesn't care about position. For instance, we have

(13).

| | |
|---|---|
| 2 | 3 |
| 1 | |

=

| | |
|---|---|
| 2 | 1 |
| 3 | |

As a $S_n$-set this is simply the left action of $S_n$ on itself, up to isomorphism.

Two $\lambda$-tableaux $s$ and $t$ are said to be *row equivalent* if the entries in each row coincide. For instance, the $(2, 1)$-tableaux

| | |
|---|---|
| 2 | 3 |
| 1 | |

and

| | |
|---|---|
| 3 | 2 |
| 1 | |

are row equivalent. Row equivalence is, clearly, an equivalence relation – its equivalence classes $\{t\}$ are called $\lambda$-tabloids.

Diagramatically, we remove the boxes from the rows to denote a $\lambda$-tabloid – the single $\lambda$-tabloid below

$$\begin{array}{cc} \hline 2 & 3 \\ \hline 1 & \\ \cline{1-1} \end{array}$$

corresponds to the two above $\lambda$-tableaux.

To see that the action of $S_n$ on the set of $\lambda$-tableaux respects row-equivalence, observe that it suffices to check it on the generators of $S_n$ – that is, the transpositions $\rho = (ij)$. Let $s \simeq t$ be row equivalent, and let $i$ and $j$ appear in rows $n$ and $m$ of $s$ respectively. Then $i$ and $j$ appear in rows $m$ and $n$ and $\rho s$ respectively. But $i$ and $j$ appear in the same rows $m$ and $n$ of $t$ as for $s$ and so likewise have their rows reversed in $\rho t$. Otherwise the two tableaux are unchanged – hence they are still row equivalent.

Because the action respects row equivalence it induces an action on the set of $\lambda$-tabloids by $\pi\{t\} = \{\pi t\}$.

**Definition 8.1.** Let $\{t_1\}, \dots, \{t_k\}$ be a complete set of $\lambda$-tabloids. We let $M^\lambda = \mathbb{C}(\{t_1\}, \dots, \{t_k\})$ denote the permutation $S_n$-module whose basis elements are the $\lambda$-tabloids.

**Examples 8.2.**
- If $\lambda = (n)$ then there is only one $\lambda$-tabloid

$$\begin{array}{ccccc} \hline 1 & 2 & 3 & \dots & \text{n} \\ \hline \end{array}$$

so $M^{(n)} = \mathbb{C}(\overline{\underline{1\ 2\ 3\dots \text{n}}})$ with the trivial action.
- If $\lambda = (1, \dots, 1)$ then no two $\lambda$-tableaux are row equivalent. In particular, a $\lambda$-tabloid just amounts to an element of $S_n$ – thus $M^\lambda = \mathbb{C}\{S_n\}$ is the regular representation – corresponding to the action of $S_n$ on $\{1, \dots, n\}$.
- In the case of $\lambda = (n-1, n)$ a $\lambda$-tabloid amounts to the choice of the single element in the second row, we denote this $\lambda$-tabloid by $\bar{i}$ – in this way we see that $M^\lambda = \mathbb{C}\{\bar{1}, \dots, \bar{n}\}$ is an isomorph of the so-called standard representation.

Note that the $M^\lambda$ above are not in general irreducible. However each contains a canonical irreducible submodule $S^\lambda \leq M^\lambda$ as we will see shortly.

**Definition 8.3.** A $G$-module $V$ is said to be cyclic if it is generated by a single element $v \in V$ – in other words each $w \in V$ is of the form $\lambda_1 g_1 v + \dots \lambda_n g_n v$ for $g_i \in G$.

**Proposition 8.4.** *Each module $M^\lambda$ is cyclic.*

*Proof.* This follows from the fact that any $\lambda$-tabloid can be taken to any other by a suitable permutation. □

**Remark 8.5.** We can describe the above more algebraically. Firstly, observe that if $\lambda \dashv n$ there is a bijection between $\lambda$-tableaux and elements of $S_n$. For instance, the tableau

$$\begin{array}{|c|c|c|}\hline 2 & 1 & 4 \\ \hline 5 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

corresponds to the permutation

$$1, 2, 3, 4, 5 \mapsto 2, 1, 4, 5, 3$$

where we read across the rows first. In particular, there are $n!$ $\lambda$-tableaux.

Under this correspondence, the action of $S_n$ on $\lambda$-tableaux corresponds to the left action of $S_n$ on itself: that is, $\pi.- : S_n \to S_n : \theta \mapsto \pi \circ \theta$.

What about row equivalence? The *Young subgroup* $S_\lambda$ is defined as the subgroup

$$S_{\{1,\ldots,\lambda_1\}} \times S_{\{\lambda_1+1,\ldots,\lambda_2\}} \times \ldots \times S_{\{n-\lambda_l+1,\ldots,n\}} \leq S_n$$

of the full symmetric group. This captures permutations which don't change rows, where the first $\lambda_1$ correspond to the first row etc. For instance, consider the row equivalent tableaux $s$ and $t$

$$\begin{array}{|c|c|c|}\hline 2 & 1 & 4 \\ \hline 5 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & 4 & 1 \\ \hline 5 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

corresponding to the permutations

$$\overline{s} : 1, 2, 3, 4, 5 \mapsto 2, 1, 4, 5, 3 \qquad \text{and} \qquad \overline{t} : 1, 2, 3, 4, 5 \mapsto 2, 4, 1, 5, 3$$

Taking the permutation $h : 1, 2, 3, 4, 5 \mapsto 1, 3, 2, 4, 5$ we see that $\overline{s}.h = \overline{t}$. In general, two tableaux $s$ and $t$ are row equivalent just when there exists $h \in S_\lambda$ such that $\overline{s}.h = \overline{t}$. In particular, the $\lambda$-tabloids $\{t_1\}, \ldots, \{t_k\}$ are thus in bijective correspondence with the left cosets $t_1S_\lambda, \ldots, t_kS_\lambda$ and the action on the former corresponds to the coset action on the latter. Passing to permutation representations, it follows that

**Proposition 8.6.** *The permutation representation* $M^\lambda = \mathbb{C}(\{t_1\}, \ldots, \{t_k\})$ *is isomorphic to the coset permutation representation* $\mathbb{C}(t_1S_\lambda, \ldots, t_kS_\lambda)$.

In particular, by Example 7.5, this means that $M^\lambda$ is the induced module of the trivial representation of the subgroup $S_\lambda$.

8.1. **Specht modules.** We now turn to the irreducible modules $S^\lambda$. To this end, we need to reconsider $\lambda$-tableaux.

**Definition 8.7.** Let $t$ be a $\lambda$-tableau.

(1) We define the row stablilser $R_t \leq S_n$ to consist of those $\pi$ which permute only the elements within each row of $t$.

(2) We define the column stabliliser $C_t \leq S_n$ to consist of those $\pi$ which permute only the elements within each column of $t$.

In particular, if $t$ has rows $R_1, \ldots, R_k$ then $R_t = S_{R_1} \times \ldots \times S_{R_k}$; if $t$ has columns $C_1, \ldots, C_k$ then $C_t = S_{C_1} \times \ldots \times S_{C_k}$.

$$\begin{array}{|c|c|c|} \hline 4 & 1 & 2 \\ \hline 3 & 5 & \\ \cline{1-2} \end{array}$$

then $R_t = S_{4,3} \times S_{4,1,2} \times S_{3,5}$ and $C_t = S_{4,3} \times S_{1,5} \times S_2$. We will be concerned primarily with column stabilisers.

But firstly, let $H \subseteq S_n$ be an arbitrary subset. We can then define an element

$$H^- = \sum_{\pi \in H} sgn(\pi)\pi \in \mathbb{C}(S_n)$$

in the group algebra. As the key instance of this, let

$$k_t = C_t^- = \sum_{\pi \in C_t} sgn(\pi)\pi \in \mathbb{C}(S_n).$$

Because $M^\lambda$ is a $S_n$-module, we obtain $k_t.- : M^\lambda \to M^\lambda$.

**Definition 8.8.** Let $t$ be a tableaux. The associated *polytabloid* is the element $e_t = k_t(\{t\}) \in M^\lambda$.

In the above example, we have $k_t = e - (4,3) - (1,5) + (4,3)(1,5)$. Hence

$$e_t = \begin{array}{c} \hline 4\ \ 1\ \ 2 \\ \hline 3\ \ 5 \\ \hline \end{array} - \begin{array}{c} \hline 3\ \ 1\ \ 2 \\ \hline 4\ \ 5 \\ \hline \end{array} - \begin{array}{c} \hline 4\ \ 5\ \ 2 \\ \hline 3\ \ 1 \\ \hline \end{array} + \begin{array}{c} \hline 4\ \ 1\ \ 2 \\ \hline 3\ \ 5 \\ \hline \end{array} .$$

**Definition 8.9.** The Specht module $S^\lambda$ is the submodule $S^\lambda = \langle e_t : t \text{ a } \lambda\text{-tableau} \rangle \leq M^\lambda$ spanned by the polytabloids, one for each $\lambda$-tableau $t$.

**Lemma 8.10.** *Let $t$ be a $\lambda$-tableau and $\pi$ a permutation. We have*
*(1)* $R_{\pi t} = \pi R_t \pi^{-1}$
*(2)* $C_{\pi t} = \pi C_t \pi^{-1}$
*(3)* $k_{\pi t} = \pi k_t \pi^{-1}$
*(4)* $e_{\pi t} = \pi e_t$.

*Proof.* These are all straightforward. For (1) we have

$$\begin{aligned} \rho \in R_{\pi t} &\iff \rho\{\pi t\} = \{\pi t\} \\ &\iff \pi^{-1}\rho\pi\{t\} = \{t\} \\ &\iff \pi^{-1}\rho\pi\{t\} \in R_t \\ &\quad\ \ \rho \in \pi R_t \pi^{-1} \end{aligned}$$

For (2) one argues in an identical fashion but using column equivalence rather than row equivalence. For (3) we have

$$k_{\pi t} = \sum_{\theta \in C_{\pi t}} sgn(\theta)\theta = \sum_{\theta \in \pi C_t \pi^{-1}} sgn(\theta)\theta = \sum_{\theta \in C_t} sgn(\pi\theta\pi^{-1})\pi\theta\pi^{-1}$$
$$= \sum_{\theta \in C_t} sgn(\theta)\pi\theta\pi^{-1} = \pi(\sum_{\theta \in C_t} sgn(\theta)\theta)\pi^{-1} = \pi k_t \pi^{-1}$$

Finally we have $\pi e_t = \pi k_t\{t\} = \pi k_t \pi^{-1}\pi\{t\} = k_{\pi t}\{\pi t\} = e_{\pi t}$. □

**Corollary 8.11.** *The Specht module $S^\lambda$ is cyclic.*

*Proof.* Since any two tableaux are related by a permutation, it follows from part (4) above that any two of the generating polytabloids are related by a permutation. The claim follows. □

**Examples 8.12.** (1) Suppose $\lambda = (n)$. Then for each $\lambda$-tableau the column stabliliser group is trivial, and we see that $k_t = \{t\}$ the unique $\lambda$-tabloid

$$\overline{\underline{\;1 \;\; 2 \;\; 3 \; \ldots \; n\;}}.$$

In particular $S^{(n)} = M^n = \mathbb{C}(\overline{\underline{1\ 2\ 3 \ldots n}})$ with the trivial module structure.

(2) Suppose $\lambda = (1, 1, \ldots, 1)$ so that there are $S_n$-many $\lambda$-tabloids $M^\lambda \cong \mathbb{C}(S_n)$. Let $t$ be the $\lambda$-tableau with vertical elements $1, 2, 3, 4, \ldots, n$. We will prove that $e_{\pi t} = sgn(\pi)e_t$ – in particular, each polytabloid is a scalar multiple of $e_t$, so that as a vector space $S^\lambda = \mathbb{C}(e_t)$. Furthermore, since we then have $\pi e_t = e_{\pi t} = sgn(\pi)e_t$ at the single basis element $e_t$ we see that $S^\lambda$ is isomorphism to the sign representation.

It remains to prove that $e_{\pi t} = sgn(\pi)e_t$. Now

$$e_{\pi t} = \pi e_t = \pi \sum_{\theta \in S_n} sgn(\theta)\theta\{t\}$$
$$= \sum_{\theta \in S_n} sgn(\theta)\pi\theta\{t\} = \sum_{\varphi \in S_n} sgn(\pi^{-1}\varphi)\pi\pi^{-1}\varphi\{t\}$$
$$= sgn(\pi^{-1}) \sum_{\varphi \in S_n} sgn(\varphi)\varphi\{t\} = sgn(\pi)e_t$$

(3) Consider $\lambda = (n-1, 1)$ again, recalling our identification $M^\lambda = \mathbb{C}(\overline{1}, \ldots, \overline{n})$. Now the $\lambda$-tableau

$$t = \begin{array}{|c|c|c|}\hline i & \ldots & j \\ \hline k & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

with $\{t\} = \overline{k}$ has column stabiliser $\{e, (ik)\}$ so that $e_t = \overline{k} - \overline{i}$. Thus $S^\lambda = \langle \overline{i} - \overline{j} : i \neq j \rangle \leq M^\lambda$. It is easy to see that this spans the subspace

$\{c_1\overline{1} + \ldots + c_n\overline{n} : \sum c_i = 0\} \leq \mathbb{C}(\overline{1}, \ldots, \overline{n})$. A basis for this subspace is given by the vectors $\overline{i} - \overline{1}$ for $i > 1$, so that the vector space has dimension $n - 1$.

This is equally the complement of the subspace $\mathbb{C}(\overline{1} + \ldots + \overline{n})$.

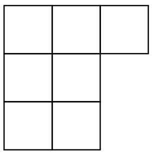
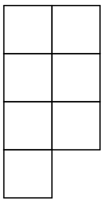
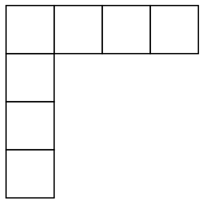
### 8.2. The dominance ordering on $\lambda$-tableaux.

**Definition 8.13.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \lambda_m)$ be partitions of $n$. We say $\lambda \trianglerighteq \mu$ (or $\lambda$ dominates $\mu$) if

$$\lambda_1 + \lambda_2 + \lambda_i \geq \mu_1 + \mu_2 + \mu_i$$

for all $i \geq 1$. (For $i \geq l$ we set $\lambda_i = 0$ and likewise if $i \geq m$ we take $\mu_i$ to be zero.)

For instance

 dominates  but is incomparable to 

**Remark 8.14.** By the above, it is not a total order, though it is in fact a lattice – we will not need this fact.

**Lemma 8.15** (Dominance lemma). *Let $\lambda, \mu \dashv n$ and and consider tableaux $t^\lambda$ and $s^\mu$. If the elements of each row of $s$ belong to distinct columns of $t$, then $\lambda \trianglerighteq \mu$.*

*Proof.* Let $i$ in row 1 of $s^\mu$, If $i$ does not appear in row 1 of $t^\mu$, then swap $i$ in $t^\lambda$ with the element $i_1$ above it in row 1 of $t^\lambda$. By assumption $i_1$ cannot belong to row 1 of $s^\mu$ – hence we produces a new tableau $t^\lambda_1$ whose first row contains all elements in the first row of $s^\mu$, and still satisfying the hypothesis of the proposition.

Now, if $i$ in row 2 of $s^\mu$ does not belong to rows 1 or 2 of $t^\lambda_1$, we permute it with the entry above it $i_2$ in the second row of $t^\lambda_1$ to obtain $t^\lambda_2$ – since $i_2$ cannot belong to the first or second row of $s^\mu$ we obtain a new $\lambda$-tableau which has the further property that its first two rows contain the elements of the first two rows of $s^\mu$.

Continuing inductively, we obtain a tableau $t^\lambda_*$, such that for all $i$ the first $i$ rows of $s^\mu$ belong to the first $i$ rows of $t^\lambda_*$. Therefore the number of elements in the first $i$ rows of $s^\mu$ must be less than or equal to the number of elements in the first $i$ rows of $t^\lambda$. In other words, for all $i$ we have $\mu_1 \ldots + \mu_i \leq \lambda_1 + \ldots + \lambda_i$, as required. □

For an example illustrating the above proof, consider

$$s^\mu = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} \quad \text{and} \quad t^\lambda = \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 3 & 2 & 5 \\ \hline\end{array}$$

Permuting the second column, we modify it to

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 & 5 \\ \hline\end{array}$$

which now has the property that each element of the first $i$ rows of $s^\mu$ belongs to the first $i$ rows of the modified $\lambda$-tableau, and so implies $\lambda \trianglerighteq \mu$.

8.3. **Irreducibility of the Specht modules.** There is a unique complex inner product on $M^\lambda$ defined on basis vectors $\{t\}$ and $\{s\}$ by $\langle \{t\}, \{s\}\rangle = \delta_{\{t\},\{s\}}$, which is linear in the first variable and conjugate linear in the second. That is, to say

$$\langle \sum_{\{t\}} \lambda_{\{t\}}\{t\}, \sum_{\{t\}} \kappa_{\{s\}}\{s\}\rangle = \sum_{\{t\},\{s\}} \lambda_{\{t\}}\overline{\kappa_{\{s\}}}\langle \{t\}, \{s\}\rangle$$

Thanks to the fact that we use conjugate linearity in the second variable, this is positive definite. It is also $S_n$-*invariant*, in the sense that

$$\langle \pi u, \pi v\rangle = \langle u, v\rangle$$

for any permutation $\pi$. This follows from the fact that $\{t\} = \{s\}$ if and only if $\pi\{t\} = \pi\{s\}$.

**Lemma 8.16** (Sign lemma)**.** *Let $H \leq S_n$ be a subgroup.*

*(1) If $\pi \in J$ then $\pi H^- = H^-\pi = sgn(\pi)H^-$.*
*(2) For $u, v \in M^\lambda$ we have $\langle H^- u, v\rangle = \langle u, H^- v\rangle$.*
*(3) Suppose that the transposition $(a, b) \in H$. Then $H^- = k(e - (a, b))$ for some $k \in \mathbb{C}(S_n)$.*
*(4) If $t$ is a tableau with $a, b$ in the same row and $(a, b) \in H$ then $H^-\{t\} = 0$.*

*Proof.* (1)

$$\pi H^- = \pi \sum_{\theta\in H} sgn(\theta)\theta = \sum_{\theta\in H} sgn(\theta)\pi\theta = \sum_{\theta} sgn(\pi^{-1}\theta)\pi\pi^{-1}\theta$$
$$= sgn(\pi)\sum_{\theta} sgn(\theta)\theta = sgn(\pi)H^-.$$

Similarly

$$H^-\pi = (\sum_{\theta\in H} sgn(\theta)\theta)\pi = \sum_{\theta\in H} sgn(\theta)\theta\pi = \sum_{\theta} sgn(\theta\pi^{-1})\theta\pi^{-1}\pi$$
$$= sgn(\pi)\sum_{\theta} sgn(\theta)\theta = sgn(\pi)H^-.$$

(2)

$$\langle H^-u, v\rangle = \langle\sum_{\theta\in H} sgn(\theta)\theta u, v\rangle = \sum_{\theta\in H} sgn(\theta)\langle\theta u, v\rangle = \sum_{\theta\in H} sgn(\theta)\langle u, \theta^{-1}v\rangle$$
$$= \langle u, \sum_{\theta\in H} sgn(\theta)\theta^{-1}v\rangle = \langle u, \sum_{\theta\in H} sgn(\theta^{-1})\theta^{-1}v\rangle = \langle u, H^-v\rangle$$

where for the third equality we use $S_n$-invariance of the inner product.

(3) Let $L = \{e, (a, v)\}$ and let $H = k_1L \cup \ldots \cup k_nL$ be a transversal. Write $K = \{k_1, \ldots, k_n\}$. Then

$$H^- = \sum_{i=1,\ldots n} (sgn(k_ie)k_ie + sgn(k_i(a, b))k_i(a, b))$$
$$= \sum_{i=1,\ldots n} (sgn(k_i)k_i - sgn(k_i)k_i(a, b)) = K^-(e - (a, b))$$

as claimed.

(4) The assumption gives $(a, b)\{t\} = \{t\}$. Gence $H^-\{t\} = K^-(e - (a, b))\{t\} = K^-(e\{t\} - (a, b)\{t\}) = K^-(\{t\} - \{t\}) = 0$, using the preceding part.

□

**Corollary 8.17.** *Let $t = t^\lambda$ and $s = s^\mu$ be tableaux, with $\lambda, \mu \dashv n$. If $k_t\{s\} \neq 0$ then $\lambda \trianglerighteq \mu$. If, moreover, $\lambda = \mu$ then $k_t\{s\} =^+_- e_t$.*

*Proof.* Suppose $a$ and $b$ belong to the same row of $s$; if they belong to the same column of $t$, then $(a, b) \in C_t$ so that $k_t(s) = 0$ by Part 4 of the sign lemma. Therefore, by the Dominance Lemma, we have $\lambda \trianglerighteq \mu$.

We know, from the first part, that elements of each row of $s$ belong to different columns of $t$. Then, by the construction in the dominance lemma, we can find $\pi \in C_t$ so that each element of the first $i$ rows of $s$ belongs to the first $i$ rows of $\pi t$. Since the tableaux have the same shape, this means that $\pi t$ and $s$ are row equivalent – that is $\{s\} = \pi\{t\}$. By the above, we then have $k_t\{s\} = k_t\{\pi t\} = sgn\pi k_t\{t\} =^+_- e_t$. □

**Corollary 8.18.** *Let $u \in M^\lambda$ and $t$ a $\lambda$-tableau. Then $k_t(u)$ is a scalar multiple of $e_t$.*

*Proof.* We can write $u = \sum_i a_i\{s_i\}$ for $\lambda$-tableaux $s_i$. Then $k_t(u) = \sum_i a_ik_t\{s_i\} = \sum_i a_i{}^+_-e_t$, by the preceding part. □

**Theorem 8.19.** *Let $U$ be a submodule of $M^\mu$. Then $S^\mu \subseteq U$ or $U \subseteq (S^\mu)^\perp$. In particular, the $S^\mu$ are irreducible.*

*Proof.* Let $u \in U$ and consider a $\mu$-tableau $t$. Consider $k_t \in \mathbb{C}(S_n)$. Since $U$ is a submodule $k_t(u) \in U$, whilst the preceding result $k_t(u) = ae_t$ for some $a$. If $a \neq 0$ then $e_t = a^{-1}k_t(u) \in U$. Since $S^\mu$ is cyclic, with $S^\mu = \langle e_t \rangle$ we then have $S^\mu \leq U$.

The other possibility is that for all $\mu$-tableaux and $u \in U$ we have $k_t(u) = 0$. We must show that $u \in (S^\mu)^\perp$ – it suffices to show that $\langle u, e_t \rangle = 0$ for all $\mu$-tableaux $t$. Now

$$\langle u, e_t \rangle = \langle u, k_t\{t\} \rangle = \langle k_t u, \{t\} \rangle = \langle 0, \{t\} \rangle = 0$$

where the second equality uses the third part of the sign lemma.

Now suppose $U \subseteq S^\mu$. If $S^\mu \subseteq U$ then $U = S^\mu$; the other possibility is that $U \subseteq (S^\mu)^\perp$ so that $U \subseteq (S^\mu)^\perp \cap S^\mu$. Since $\langle -, - \rangle$ is positive-definite, this intersection is zero – hence $U = 0$. □

**Proposition 8.20.** *Suppose $\theta : S^\lambda \to M^\mu$ is a non-zero $S_n$-module homomorphisms. Then $\lambda \trianglerighteq \mu$, and if $\lambda = \mu$ then $\theta$ is multiplication by a scalar.*

*Proof.* Since $\theta \neq 0$ there exists a basis vector $e_t$ such that $\theta(e_t) \neq 0$. Now since $M^\lambda = S^\lambda \oplus (S^\lambda)^\perp$ we can extend this to a non-zero homomorphism $\theta : M^\lambda \to M^\mu$.

Now $0 \neq \theta(e_t) = \theta(k_t\{t\} = k_t\theta(\{t\}) = k_t(\sum_i c_i\{s_i\} = \sum_i c_i k_t\{s_i\}$, so there exists some $\mu$-tableau $k_t(\{s_i\}) \neq 0$. Therefore $\lambda \trianglerighteq \mu$.

Finally, by Corollary 8.18, we have that $\theta(e_t) = k_t\theta(\{t\}) = ce_t$ for some non-zero scalar $c$. Then $\theta(e_{\pi t}) = \theta(\pi e_t) = \pi\theta(e_t) = \pi c e_t = c\pi e_t$. Hence $\theta$ is multiplication by $c$. □

**Theorem 8.21.** *The $S^\lambda$ for $\lambda \dashv n$ form a complete list of irreducible $S_n$-modules.*

*Proof.* We have already proven that they are irreducible. Let us prove that they are pair-wise non-isomorphic. Suppose $\lambda \neq \mu$ but $\theta : S^\lambda \cong S^\mu$. Then, by composition, we obtain an injective homomorphism $S^\lambda \to M^\mu$ and likewise $S^\mu \to M^\lambda$. By the preceding result we then have $\lambda \trianglerighteq \mu$ and $\mu \trianglerighteq \lambda$ so that $\lambda = \mu$ – a contradiction.

Since the number of partitions equals the conjugacy classes of $S_n$, this gives a complete list of the irreducible representations of $S_n$. □

We have not said anything so far about the bases or dimensions of these irreducible representations. At this point, I will just give a brief survey of this topic.

**Definition 8.22.** A tableau is said to be *standard* if both its rows and columns form increasing sequences.

For example,

| | | |
|---|---|---|
| 1 | 2 | 6 |
| 3 | 4 | |
| 5 | | |

is standard but

| | | |
|---|---|---|
| 1 | 2 | 6 |
| 4 | 3 | |
| 5 | | |

is not.

**Theorem 8.23.** *The set* $\{e_t : t \text{ is a standard } \lambda\text{-tableau}\}$ *forms a basis for* $S^\lambda$.

We will not prove the above, or anything else about these representations, though there is much more to say.

Let $f^\lambda$ denote the number of standard $\lambda$-tableaux. Then since the number of elements of a group equals the sum of the squares of the dimensions of its irreducible representations, we have as an immediate consequence of the above theorem

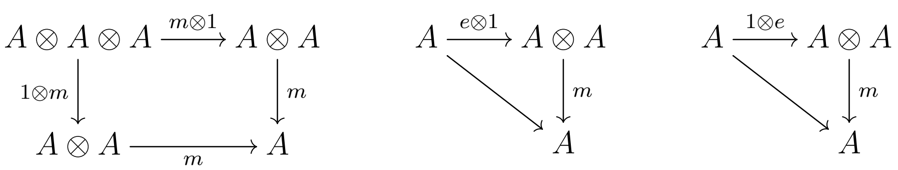
**Corollary 8.24.** $n! = \sum_{\lambda:\lambda\dashv n}(f_\lambda)^2$.

This is a purely combinatorial identity which admits other interesting proofs. An elegant proof is via the so-called Robinson-Schensted correspondence, which draws a bijective correspondence between pairs of standard tableaux and elements of $S_n$.

## 9. Representations of Hopf algebras

Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category – for example, the category $(\mathbf{Set}, \times, 1)$ with cartesian product, the category $(Ab, \otimes, \mathbb{Z})$ with tensor product of abelian groups or $(\mathbf{Vect}, \otimes, k)$ with the tensor product of vector spaces.

A monoid/algebra in $\mathcal{C}$ is an object $A$ equipped with morphisms $m : A \otimes A \to A$ and $e : l \to A$ satisfying the associativity:




It is said to be commutative if $m = m \circ s : A \otimes A \to A \otimes A \to A$ where $s$ is the symmetry.

**Remark 9.1.** We write as if we are in a strict monoidal category, rather than a free one, for notational convenience.

**Examples 9.2.** An algebra in **Set** is a monoid. An algebra in $(Ab, \otimes, \mathbb{Z})$ is a ring. An algebra in $(\mathbf{Vect}, \otimes, k)$ is an algebra in the classical sense – that is, an associative unital algebra.

An algebra morphism $f : (A, m, e) \to (B, m', e')$ is a morphism $f : A \to B$ such that

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ A & \xrightarrow[f]{} & B \end{array} \qquad \begin{array}{ccc} I & & \\ {\scriptstyle e}\downarrow & \searrow^{e'} & \\ A & \xrightarrow[f]{} & B \end{array}$$

Algebras and algebra morphisms form a category $U : Alg(\mathcal{C}) \to \mathcal{C}$ over $\mathcal{C}$.

A coalgebra in $\mathcal{C}$ is an object $A$ equipped with a comultiplication $\Delta : A \to A \otimes A$ and counit $q : A \to I$ satisfying the coassociativity and counit equations dual to the above ones. Equivalently, it is an algebra in $(\mathcal{C}^{op}, \otimes, I)$. Again we have the category $Coalg(\mathcal{C}) \to \mathcal{C}$.

**Examples 9.3.** In **Set**, or any cartesian monoidal category, each object $X$ admits a unique coalgebra structure with comultiplication $\Delta : X \to X \times X : x \mapsto (x, x)$ and counit $X \to 1$ the unique such map. Good exercise: check the details of this!

A *bialgebra* is an object $A$ equipped with both an algebra structure $(A, m, e)$ and a coalgebra structure $(A, \Delta, q)$ satisfying the further equations

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\Delta \otimes \Delta} & A \otimes A \otimes A \otimes A \\ & & \downarrow{\scriptstyle 1 \otimes s \otimes 1} \\ {\scriptstyle m}\downarrow & & A \otimes A \otimes A \otimes A \\ & & \downarrow{\scriptstyle m \otimes m} \\ A & \xrightarrow{\Delta} & A \otimes A \end{array} \qquad \begin{array}{ccc} I & & \\ {\scriptstyle e}\downarrow & \searrow^{e \otimes e} & \\ A & \xrightarrow{\Delta} & A \otimes A \end{array}$$

and

$$\begin{array}{ccc} I & \xrightarrow{e} & A \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle q} \\ & & I \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{m} & A \\ & {\scriptstyle q \otimes q}\searrow & \downarrow{\scriptstyle q} \\ & & I \end{array}$$

**Remark 9.4.** Note that since $\mathcal{C}$ is symmetric monoidal, the categories $Alg(\mathcal{C})$ and $Coalg(\mathcal{C})$ inherit symmetric monoidal structure lifted from $\mathcal{C}$. Given algebras $A$ and $B$ we form $A \otimes B$ and equip it with the tensor product structure

$$A \otimes A \otimes B \otimes B \xrightarrow{1 \otimes s \otimes 1} A \otimes B \otimes A \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B$$

and

$$I \xrightarrow{e_A \otimes e_B} A \otimes B$$

Note that this does not work at all if $\mathcal{C}$ is not symmetric. Observe also that it gives the usual pointwise formula for product of monoids when $\mathcal{C} = \mathbf{Set}$ – that is, $(a, b) \star (a', b') = (m_A(a, a'), m_B(b, b'))$.

Checking the axioms and lifting the remainder of the structure is straightforward – in particular, the forgetful functors $Alg(\mathcal{C}), Coalg(\mathcal{C}) \to \mathcal{C}$ are *strict* monoidal.

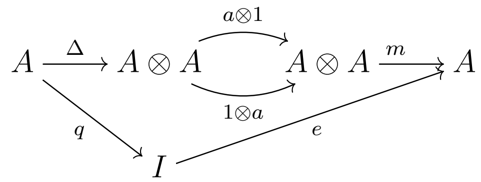
The first two equations of a bialgebra say that $\Delta : A \to A \otimes A$ is a morphism of algebras and the second two that $q : A \to I$ is a morphism of algebras. In other words, a bialgebra is exactly a coalgebra in the category of algebras! Reading the diagrams vertically, we see that it is equally an algebra in the category of coalgebras. Indeed, we have

$$Bialg(\mathcal{C}) \cong Coalg(Alg(\mathcal{C})) \cong Alg(Coalg(\mathcal{C})) \quad .$$

A Hopf algebra is a bialgebra $A$ further equipped with an *antipode map*

$$a : A \to A$$

such that the two composites on the top row equal that on the bottom below




Here is another way of looking at the definition of the antipode. I will just explain it in the case $\mathcal{C} = \mathbf{Vect}$ since my goal here is to make it intuitive, although it is easily seen to be true in any symmetric monoidal *closed* category.

Firstly, if $A$ a coalgebra and $B$ an algebra, then the hom vector space $\mathbf{Vect}(A, B)$ becomes an *algebra* if at $f, g : A \rightrightarrows B$ we define

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{f \otimes g} B \otimes B \xrightarrow{m} B$$

and the unit element to be $e \circ q : A \to k \to A$. This is called the *convolution algebra*.

In the case that $A = B$, we see that the antipode is exactly an element of the convolution algebra $[A, A]$ which is an inverse for $1 : A \to A$ under the convolution product – that is, a *convolution inverse of* $1 : A \to A$.

Since inverses in an algebra are unique, observe that this shows that antipodes are *unique*, if they exist.

**Examples 9.5.** (1) A Hopf algebra in $(\mathbf{Set}, \times, 1)$ is precisely a group. The map $a : A \to A$ sends an element to an inverse – the two above equations indeed say exactly this: that is, $m(a(x), x) = e = m(x, a(x))$.

(2) A Hopf algebra in **Vect** is, by definition, a *Hopf algebra.* We will now show that the group algebra $k[G]$ is a Hopf algebra. Here is an abstract explanation. The functor $k[-] : \mathbf{Set} \to \mathbf{Vect}$ sending a set $X$ to the free vector space $k[X]$ is a strong symmetric monoidal functor – in other words, we have isomorphisms $k[X] \otimes k[Y] \cong k[X \times Y]$ and $k \cong k[1]$ compatible with the associativity isomorphisms, unit and symmetry isomorphisms. Any such functor sends algebras to algebras, coalgebras to coalgebras, bialgebras to bialgebras and Hopf algebras to Hopf algebras in an obvious way! Since Hopf algebras in **Set** are exactly groups, it follows that if $G$ is a group then the group algebra $k[G]$ is a *Hopf algebra.*

That is, we have algebra structure

$$k[G] \otimes k[G] \xrightarrow{\cong} k[G \times G] \xrightarrow{k[m]} k[G] \qquad k \xrightarrow{\cong} k[1] \xrightarrow{k[e]} k[G]$$

and coalgebra structure

$$k[G] \xrightarrow{k[\Delta]} k[G \times G] \xrightarrow{\cong} k[G] \otimes k[G] \qquad k[G] \xrightarrow{k[!]} k[1] \xrightarrow{\cong} k$$

with antipode

$$k[a] : k[G] \to k[G].$$

The algebra structure is that which we have seen many times in the course. The coalgebra structure is given by $\Delta_{k[G]}(\Sigma_i \lambda_i g_i) = \Sigma_i \lambda_i (g_i \otimes g_i)$ and $q_{k[G]}(\Sigma_i \lambda_i g_i) = \Sigma_i \lambda_i$, whilst the antipode $a_{k[G]} : k[G] \to k[G]$ similarly sends $\Sigma_i \lambda_i g_i$ to $\Sigma_i \lambda_i (g_i)^{-1}$. Since the group $G$ is a cocommutative bialgebra in $Set$ – i.e. the comultiplication $\Delta$ is cocommutative – it follows that $k[G]$ is also a cocommutative bialgebra. Of course it won't be commutative unless $G$ is abelian.

(3) It would be nice to add further examples here, such as the universal enveloping algebra of a Lie algebra, the tensor algebra of a vector space and some combinatorial examples.

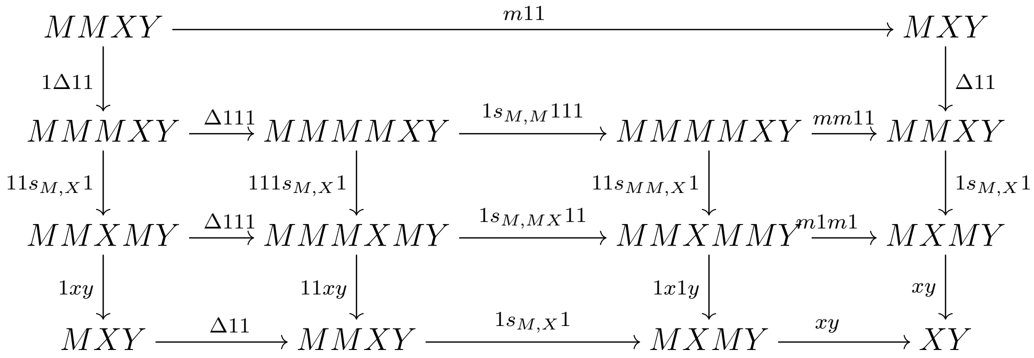
9.1. **Modules and comodules over a bialgebra.** Since a bialgebra is both an algebra and a coalgebra we can consider both its modules and its comodules. Here, we will focus on modules since $k[G]$-modules are precisely representations of $G$ – see Section 1.3.

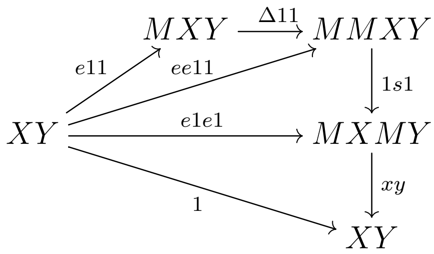
Let $M$ be a bialgebra, and $X, Y$ (left) M-modules, via actions $x : M \otimes X \to X$ and $y : M \otimes Y \to Y$. Then $X \otimes Y$ is a $M$-module with action:

$$M \otimes X \otimes Y \xrightarrow{\Delta \otimes 1 \otimes 1} M \otimes M \otimes X \otimes Y \xrightarrow{1 \otimes s \otimes 1} M \otimes X \otimes M \otimes Y \xrightarrow{a \otimes b} X \otimes Y$$

Observe, that is generalises the tensor products of $G$-modules when $M = k[G]$ since it just captures the formula $g.(a \otimes b) = ga \otimes gb$.

Here are the associativity and unit equations for the module.





In the above, we use that $\Delta$ is an algebra map – once in the first and once in the second diagram. Otherwise, it is straightforward to see that the tensor product lifts to give a monoidal structure on $M-Mod$ preserved by the forgetful functor to $\mathcal{C}$. (If one wants to lift the symmetry to $M-Mod$ then one requires that the bialgebra be cocommutative.)

In fact, for $\mathcal{C} = \mathbf{Vect}$ there is a *bijection* between

- Bialgebra structures on the $k$-algebra $M$.
- Monoidal structures on $M-Mod$ strictly preserved by $U : M-Mod \to \mathcal{C}$

In the opposite direction, if we have a lifted monoidal structure $(M-Mod, \otimes, I)$ how do we get the bialgebra structure. It is easy to see that $(M, m)$ is itself the free $M$-module on the unit $I$; hence we can form the tensor product $M$-module $(M, m) \otimes (M, m) = (M \otimes M, n)$ where we write $n$ the left action $n : M \otimes M \otimes M \to M \otimes M$; then the comultiplication for the coalgebra $M$ is given by the composite

$$M \xrightarrow{1ee} MMM \xrightarrow{n} MM.$$

For the counit, we use that $I$ lifts to an $M$-module; its actions then gives the counit $M \cong M \otimes I \cong I$ for the coalgebra structure.

**Remark 9.6.** Here is a more general approach to the above. An opmonoidal functor $(F, f, f_0) : A \rightsquigarrow B$ consists of a functor $F : A \to B$, coherence constraints $f_{a,b} : F(ab) \to FaFb$ natural in $a$ and $b$ and a comparison $f_0 : Fi \to i$ compatible with the monoidal structure – that is, the associativity and unit isomorphisms. Opmonoidal functors can be composed – given $F : A \to B$ and $G : B \to C$ as above, the components

$$GF(ab) \xrightarrow{Gf} G(FaFb) \xrightarrow{g} GFaGFb \qquad\qquad GFi \xrightarrow{Gf_0} Gi \xrightarrow{g_0} i$$

equip $GF$ with opmonoidal structure. A monoidal transformation between opmonoidal functors is a natural transformation satisfying the extra conditions

$$\begin{array}{ccc} F(ab) & \xrightarrow{\eta_{ab}} & G(ab) \\ {\scriptstyle f_{a,b}}\downarrow & & \downarrow{\scriptstyle g_{a,b}} \\ FaFb & \xrightarrow[\eta_a\eta_b]{} & GaGb \end{array} \qquad \begin{array}{ccc} i^B & \xrightarrow{f_0} & Fi^A \\ & {\scriptstyle g_0}\searrow & \downarrow{\scriptstyle \eta_i} \\ & & Gi^A \end{array}$$

which we call the *associativity, left unit* and *right unit* conditions.

An *opmonoidal monad* is a monad $(T, \eta, \mu)$ on a monoidal category $\mathcal{C}$ such that $T$ is a opmonoidal functor and $\eta, \mu$ opmonoidal transformations. Now, if $A$ is an algebra then $A \otimes - : \mathcal{C} \to \mathcal{C}$ is a monad, whose multiplication and unit have components $m \otimes 1 : A \otimes A \otimes X \to A \otimes X$ and unit $e \otimes 1 : X \to A \otimes A$. The algebras for the monad are then precisely the left $A$-modules.

If $A$ is furthermore, a bialgebra, then $A \otimes -$ is an opmonoidal monad – henceforth I will use juxtaposition for tensor product. The structure maps are

$$AXY \xrightarrow{\Delta 11} AAXY \xrightarrow{1s1} AXAY \qquad A \xrightarrow{q} I$$

It is easy to see that this does give an opmonoidal functor, using that $A$ is a coalgebra. The two equations that we proved above (if we ignore the bottom rows) describe the interaction of $MXY \to MXMY$ and $XY \to MXY$ with muliplication and unit respectively, and the first is by far the trickiest to check.

If $T$ is an opmonoidal monad then the category of algebras Alg$T$ admits a monoidal structure such that the forgetful functor $U : \mathrm{Alg}T \to \mathcal{C}$ is strict monoidal. Given $T$-algebras $(X, x)$ and $(Y, y)$ the tensor $T$-algebra on $XY$ has structure map

$$T(XY) \xrightarrow{\varphi} TXTY \xrightarrow{xy} XY$$

where $\varphi$ is the component of the opmonoidal functor. This generalises the tensor product of modules described above. In fact, for general monad $T$ on $\mathcal{C}$ there is a *bijection* between

- opmonoidal structures on the monad $T$.
- Monoidal structures on $AlgT$ strictly preserved by $U : AlgT \to \mathcal{C}$

9.2. **Antipodes and internal homs.** In the case that $M$ is a Hopf algebra we can say a bit more – namely, we can show that the internal hom in $\mathcal{C}$ *also* lifts to the category of modules. The main idea is this:

(1) We have seen that $\Delta : M \to M \otimes M$ is an algebra map.

(2) Now we can also form the opposite algebra $M^{op}$ with reversed multiplication $m \circ s : M \otimes M \to M \otimes M \to M$. It turns out that if $M$ is a Hopf algebra then the antipode is an algebra map $a : M \to M^{op}$; note that for groups this captures the fact that $(xy)^{-1} = y^{-1}x^{-1}$.
(3) Now, in general, if $X, Y$ are left $M$-modules then the hom $[X, Y]$ is an $M$-bimodule, or equally, an $M^{op} \otimes M$-module. The point is that the action is contravariant in $X$ and covariant in $Y$ – it is given by $(a \otimes b.f)x = bf(ax)$.
(4) Now we can restrict scalars along the algebra map $(a \otimes 1) \circ \Delta : M \to M \otimes M \to M^{op} \otimes M$ to equip $[X, Y]$ with the structure of a left $H$-module.
(5) Explicitly, we can give the formula as follows:

$$M \otimes [X,Y] \xrightarrow{\Delta\otimes 1} M \otimes M \otimes [X,Y] \xrightarrow{1\otimes s} M \otimes [X,Y] \otimes M \xrightarrow{\mu_1\otimes a} [X,Y] \otimes M \xrightarrow{\mu_2} [X,Y]$$

where $\mu_1$ is the left action

$$M \otimes [X,Y] \xrightarrow{\tilde{y}\otimes 1} [Y,Y] \otimes [X,Y] \xrightarrow{\circ} [X,Y]$$

sending $m \otimes f$ to $x \mapsto mf(x)$), and $\mu_2$ the contravariant action

$$[X,Y] \otimes M \xrightarrow{1\otimes \tilde{x}} [X,Y] \otimes [X,X] \xrightarrow{\circ} [X,Y]$$

sending $m \otimes f$ to $x \mapsto f(mx)$).

For groups representations, this gives the usual formula for the internal hom

$$(g.f)x = g(f(g^{-1}x)).$$

## 10. Exam

Know the main topics and results in the course and proofs of the theorems. The main topics we covered are

(1) Basics on irreducibility: Maschkes theorem, Schur's lemma, decomposition of modules into irreducibles, decomposition of the regular module, representations of abelian groups.
(2) Tensors and homs of $G$-modules: the relationship between homs, tensor products and duals.
(3) Characters, their basic properties, how characters determine a representation, orthonormality of irreducible characters, the number of irreducible characters, character tables, the inner product and orthogonality relations, calculating character tables, Frobenius reciprocity formula, but not the proof using Kan extensions.
(4) Representations of the symmetric group – definitions of tableaux, tabloids, Specht modules, dominance ordering, . . .
(5) Definitions of Hopf algebras.

The exam will involve some theoretical work – proofs etc – as well as lots of calculations, involving irreducible representable and character tables. It will not be very category theoretic. I recommend doing the assignments as practice.

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