Lecture 5: Spin

Lessons from non-relativistic quantum mechanics:

- $\langle A \rangle = \langle \Psi | A | \Psi \rangle$ expected value

- $[\hat{r}_j, \hat{p}_k] = \mathrm{i}\hbar \quad \Rightarrow \quad [\hat{L}_x, \hat{L}_y] = \mathrm{i}\hbar \hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = \mathrm{i}\hbar \hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = \mathrm{i}\hbar \hat{L}_y$ commutators

- $\hat{r}_j \psi = r_j \cdot \psi, \quad \hat{p}_j \psi = -\mathrm{i}\hbar \frac{\partial}{\partial r_j} \psi$ operators of position and momentum

- $\hat{H}\psi = \mathrm{i}\hbar \frac{\partial}{\partial t}$, operator of energy, equation of motion

- free particle: $\hat{H} = (\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2)/2m$ $p^2/2m$ is kinetic energy

- electric field: $\hat{H} = (\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2)/2m + QV \quad \Rightarrow \quad \hat{H} - QV = (\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2)/2m$

- electromagnetic field: $\hat{H} - QV = ((\hat{p}_x - QA_x)^2 + (\hat{p}_y - QA_y)^2 + (\hat{p}_z - QA_z)^2)/2m$,
  $\vec{B} = \vec{\nabla} \times \vec{A}$

Lessons from relativistic quantum mechanics:

$$t = \frac{t_0}{\sqrt{1 - v^2/c^2}} \qquad m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \qquad \mathcal{E}_t = mc^2 = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}, \tag{1}$$

$$\mathcal{E}_t^2 - c^2 p_x^2 - c^2 p_y^2 - c^2 p_z^2 = m_0^2 c^4 \Rightarrow \mathcal{E}_t^2 - c^2 p_x^2 - c^2 p_y^2 - c^2 p_z^2 - m_0^2 c^4 = 0 \tag{2}$$

$$\left(\hbar^2 \frac{\partial^2}{\partial t^2} - c^2\hbar^2 \frac{\partial^2}{\partial z^2} - c^2\hbar^2 \frac{\partial^2}{\partial x^2} - c^2\hbar^2 \frac{\partial^2}{\partial y^2} + (m_0 c^2)^2\right) \Psi = 0 \tag{3}$$

not equation of motion!

$$\left(\mathrm{i}\hbar \frac{\partial}{\partial t}\widehat{\gamma}^0 + \mathrm{i}c\hbar \frac{\partial}{\partial x}\widehat{\gamma}^1 + \mathrm{i}c\hbar \frac{\partial}{\partial y}\widehat{\gamma}^2 + \mathrm{i}c\hbar \frac{\partial}{\partial z}\widehat{\gamma}^3 - m_0 c^2 \widehat{1}\right) \Psi = 0, \tag{4}$$

equation of motion

electromagnetic field: $\hat{H} - QV = ((\hat{p}_x - QA_x)^2 + (\hat{p}_y - QA_y)^2 + (\hat{p}_z - QA_z)^2)/2m$,
$\vec{B} = \vec{\nabla} \times \vec{A}$

$$\begin{aligned}
&\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - QV\right)\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - QV\right)\hat{1}\Psi = \left(\mathrm{i}\hbar\frac{\partial}{\partial t} - QV\right)^2\Psi = \\
&\left(c^2\left(\mathrm{i}\hbar\frac{\partial}{\partial x} + QA_x\right)^2\hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^0\hat{\gamma}^1 + c^2\left(\mathrm{i}\hbar\frac{\partial}{\partial y} + QA_y\right)^2\hat{\gamma}^0\hat{\gamma}^2\hat{\gamma}^0\hat{\gamma}^2 + c^2\left(\mathrm{i}\hbar\frac{\partial}{\partial z} + QA_z\right)^2\hat{\gamma}^0\hat{\gamma}^3\hat{\gamma}^0\hat{\gamma}^3\right)\Psi \\
&+m_0^2c^4\hat{\gamma}^0\hat{\gamma}^0\Psi \\
&-m_0c^3\left(\left(\mathrm{i}\hbar\frac{\partial}{\partial x} + QA_x\right)\hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^0 + \left(\mathrm{i}\hbar\frac{\partial}{\partial y} + QA_y\right)\hat{\gamma}^0\hat{\gamma}^2\hat{\gamma}^0 + \left(\mathrm{i}\hbar\frac{\partial}{\partial z} + QA_z\right)\hat{\gamma}^0\hat{\gamma}^3\hat{\gamma}^0\right)\Psi \\
&-m_0c^3\left(\left(\mathrm{i}\hbar\frac{\partial}{\partial x} + QA_x\right)\hat{\gamma}^0\hat{\gamma}^0\hat{\gamma}^1 + \left(\mathrm{i}\hbar\frac{\partial}{\partial y} + QA_y\right)\hat{\gamma}^0\hat{\gamma}^0\hat{\gamma}^2 + \left(\mathrm{i}\hbar\frac{\partial}{\partial z} + QA_z\right)\hat{\gamma}^0\hat{\gamma}^0\hat{\gamma}^3\right)\Psi \\
&+c^2\left(\left(\mathrm{i}\hbar\frac{\partial}{\partial x} + QA_x\right)\left(\mathrm{i}\hbar\frac{\partial}{\partial y} + QA_y\right)\hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^0\hat{\gamma}^2 + \left(\mathrm{i}\hbar\frac{\partial}{\partial y} + QA_y\right)\left(\mathrm{i}\hbar\frac{\partial}{\partial x} + QA_x\right)\hat{\gamma}^0\hat{\gamma}^2\hat{\gamma}^0\hat{\gamma}^1\right)\Psi \\
&+c^2\left(\left(\mathrm{i}\hbar\frac{\partial}{\partial y} + QA_y\right)\left(\mathrm{i}\hbar\frac{\partial}{\partial z} + QA_z\right)\hat{\gamma}^0\hat{\gamma}^2\hat{\gamma}^0\hat{\gamma}^3 + \left(\mathrm{i}\hbar\frac{\partial}{\partial z} + QA_z\right)\left(\mathrm{i}\hbar\frac{\partial}{\partial y} + QA_y\right)\hat{\gamma}^0\hat{\gamma}^3\hat{\gamma}^0\hat{\gamma}^2\right)\Psi \\
&+c^2\left(\left(\mathrm{i}\hbar\frac{\partial}{\partial z} + QA_z\right)\left(\mathrm{i}\hbar\frac{\partial}{\partial x} + QA_x\right)\hat{\gamma}^0\hat{\gamma}^3\hat{\gamma}^0\hat{\gamma}^1 + \left(\mathrm{i}\hbar\frac{\partial}{\partial x} + QA_x\right)\left(\mathrm{i}\hbar\frac{\partial}{\partial z} + QA_z\right)\hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^0\hat{\gamma}^3\right)\Psi
\end{aligned} \tag{5}$$

HOMEWORK!

$$\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - QV\right)^2 \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{1} \end{pmatrix} \Psi = \tag{6}$$

$$\left(c^2\left(\mathrm{i}\hbar\frac{\partial}{\partial x} + QA_x\right)^2 + c^2\left(\mathrm{i}\hbar\frac{\partial}{\partial y} + QA_y\right)^2 + c^2\left(\mathrm{i}\hbar\frac{\partial}{\partial z} + QA_z\right)^2 + m_0^2c^4\right) \begin{pmatrix} \hat{1} & \hat{0} \\ \hat{0} & \hat{1} \end{pmatrix} \Psi \tag{7}$$

$$-c^2\hbar Q\left(B_x \begin{pmatrix} \hat{\sigma}^1 & \hat{0} \\ \hat{0} & \hat{\sigma}^1 \end{pmatrix} + B_y \begin{pmatrix} \hat{\sigma}^2 & \hat{0} \\ \hat{0} & \hat{\sigma}^2 \end{pmatrix} + B_z \begin{pmatrix} \hat{\sigma}^3 & \hat{0} \\ \hat{0} & \hat{\sigma}^3 \end{pmatrix}\right)\Psi \tag{8}$$

Approximation for low speed:

$$(m_0c^2)^2 = \mathcal{E}_t^2 - c^2p^2 = (m_0c^2 + \mathcal{E})^2 - c^2p^2 = (m_0c^2)^2 + 2\mathcal{E}(m_0c^2) + \mathcal{E}^2 - c^2p^2 \tag{9}$$

free particle: $\mathcal{E}$ is kinetic energy, at low speed: $\mathcal{E} \ll m_0c^2$

$$(m_0c^2)^2 \approx (m_0c^2)^2 + 2\mathcal{E}(m_0c^2) - c^2p^2 \tag{10}$$

$$\mathcal{E} = \frac{p^2}{2m_0} \tag{11}$$

Approximation for low speed:

$$\widehat{H} \approx \frac{1}{2m_0} \left(\left(\mathrm{i}\hbar \frac{\partial}{\partial x} + QA_x \right)^2 + \left(\mathrm{i}\hbar \frac{\partial}{\partial y} + QA_y \right)^2 + \left(\mathrm{i}\hbar \frac{\partial}{\partial z} + QA_z \right)^2 + QV \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{12}$$

$$-\frac{\hbar Q}{2m_0} \left(B_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + B_y \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} + B_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \tag{13}$$

$$\mathcal{E} = -\vec{\mu} \cdot \vec{B} = -(\mu_x B_x + \mu_y B_y + \mu_z B_z) \tag{14}$$

$$\widehat{H} = -\left(\frac{\hbar Q}{2m_0} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B_x + \frac{\hbar Q}{2m_0} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} B_y + \frac{\hbar Q}{2m_0} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} B_z \right) \tag{15}$$

$$\widehat{I}_x\widehat{I}_y - \widehat{I}_y\widehat{I}_x = \mathrm{i}\hbar\widehat{I}_z, \qquad \widehat{I}_y\widehat{I}_z - \widehat{I}_z\widehat{I}_y = \mathrm{i}\hbar\widehat{I}_x, \qquad \widehat{I}_z\widehat{I}_x - \widehat{I}_x\widehat{I}_z = \mathrm{i}\hbar\widehat{I}_y. \tag{16}$$

Guess:

$$\gamma = \frac{Q}{2m} \qquad \Rightarrow \qquad \widehat{I}_x = \hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \widehat{I}_y = \hbar\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \widehat{I}_z = \hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{17}$$

Check:

$$\widehat{I}_x\widehat{I}_y - \widehat{I}_y\widehat{I}_x = \hbar^2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} - \hbar^2\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tag{18}$$

$$\hbar^2\left(\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} - \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}\right) = 2\mathrm{i}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2\mathrm{i}\hbar\widehat{I}_z \tag{19}$$

$$\widehat{I}_x\widehat{I}_y - \widehat{I}_y\widehat{I}_x = \mathsf{i}\hbar\widehat{I}_z, \qquad \widehat{I}_y\widehat{I}_z - \widehat{I}_z\widehat{I}_y = \mathsf{i}\hbar\widehat{I}_x, \qquad \widehat{I}_z\widehat{I}_x - \widehat{I}_x\widehat{I}_z = \mathsf{i}\hbar\widehat{I}_y. \tag{20}$$

New guess:

$$\gamma = \frac{Q}{m} \qquad \Rightarrow \qquad \widehat{I}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \widehat{I}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -\mathsf{i} \\ \mathsf{i} & 0 \end{pmatrix} \qquad \widehat{I}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{21}$$

Check:

$$\widehat{I}_x\widehat{I}_y - \widehat{I}_y\widehat{I}_x = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathsf{i} \\ \mathsf{i} & 0 \end{pmatrix} - \frac{\hbar^2}{4}\begin{pmatrix} 0 & -\mathsf{i} \\ \mathsf{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tag{22}$$

$$\frac{\hbar^2}{4}\left(\begin{pmatrix} \mathsf{i} & 0 \\ 0 & -\mathsf{i} \end{pmatrix} - \begin{pmatrix} -\mathsf{i} & 0 \\ 0 & \mathsf{i} \end{pmatrix}\right) = \mathsf{i}\frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathsf{i}\hbar\widehat{I}_z \tag{23}$$

$$\text{Operator}\text{Eigenfunction} = \text{Eigenvalue} \cdot \text{Eigenfunction} \tag{24}$$

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sqrt{\frac{1}{h^3}}\begin{pmatrix} \psi \\ 0 \end{pmatrix} \quad = \quad \frac{\hbar}{2} \quad \sqrt{\frac{1}{h^3}}\begin{pmatrix} \psi \\ 0 \end{pmatrix} \tag{25}$$

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sqrt{\frac{1}{h^3}}\begin{pmatrix} 0 \\ \psi \end{pmatrix} \quad = \quad -\frac{\hbar}{2} \quad \sqrt{\frac{1}{h^3}}\begin{pmatrix} 0 \\ \psi \end{pmatrix} \tag{26}$$

$$\text{Operator}\,\text{Eigenfunction} = \text{Eigenvalue} \cdot \text{Eigenfunction} \tag{27}$$

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sqrt{\frac{1}{h^3}}\psi \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad = \quad \frac{\hbar}{2} \quad \sqrt{\frac{1}{h^3}}\psi \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{28}$$

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sqrt{\frac{1}{h^3}}\psi \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad = \quad -\frac{\hbar}{2} \quad \sqrt{\frac{1}{h^3}}\psi \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{29}$$

$$\text{Operator}\,\text{Eigenvector} = \text{Eigenvalue} \cdot \text{Eigenvector} \tag{30}$$

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{31}$$

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{32}$$

- If the particle is in state $|\alpha\rangle$, the result of measuring I_z is *always* $+\hbar/2$. The expected value is

$$\langle I_z\rangle = \langle\alpha|I_z|\alpha\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{\hbar}{2}. \tag{33}$$

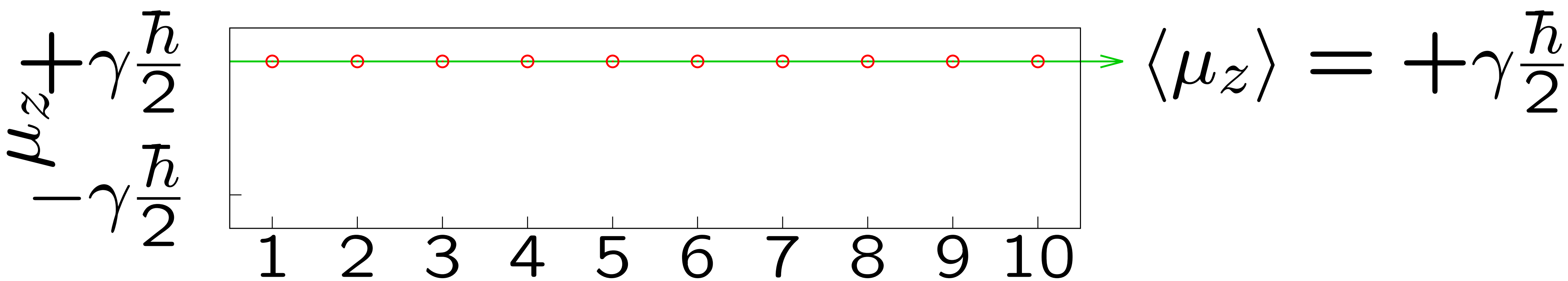
- If the particle is in state $|\beta\rangle$, the result of measuring I_z is *always* $-\hbar/2$. The expected value is

$$\langle I_z\rangle = \langle\beta|I_z|\beta\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}. \tag{34}$$

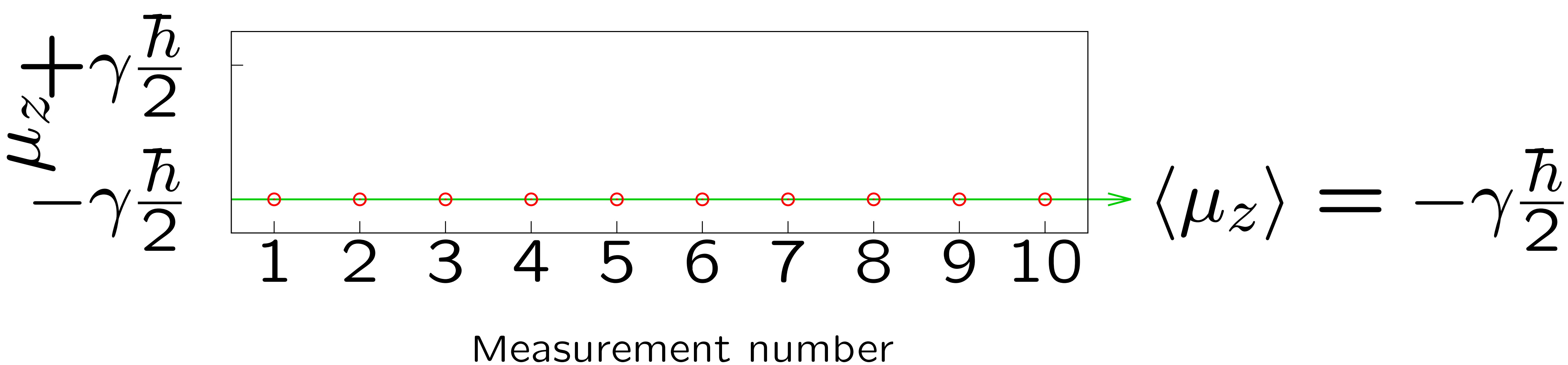
- Any state $c_\alpha|\alpha\rangle + c_\beta|\beta\rangle$ is possible, but the result of a single measurement of I_z is *always* $+\hbar/2$ or $-\hbar/2$. However, the expected value of I_z is

$$\langle I_z\rangle = \langle\alpha|I_z|\beta\rangle = \begin{pmatrix} c_\alpha^* & c_\beta^* \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_\alpha \\ c_\beta \end{pmatrix} = (|c_\alpha|^2 - |c_\beta|^2)\frac{\hbar}{2}. \tag{35}$$

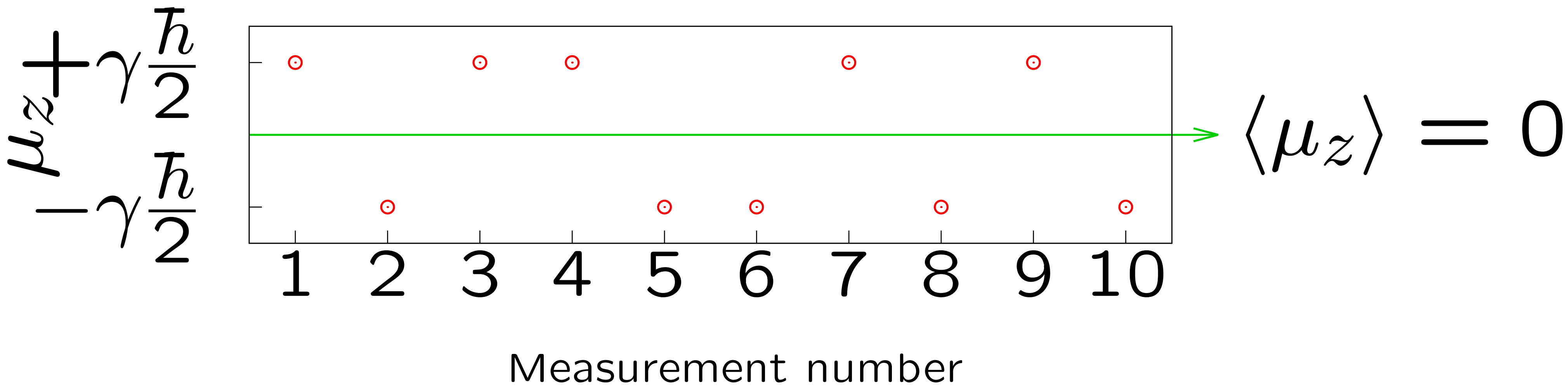
$|\Psi\rangle = |\alpha\rangle$



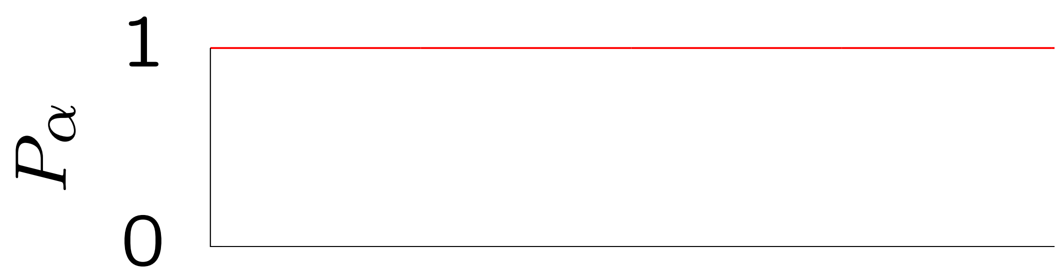
$|\Psi\rangle = |\beta\rangle$



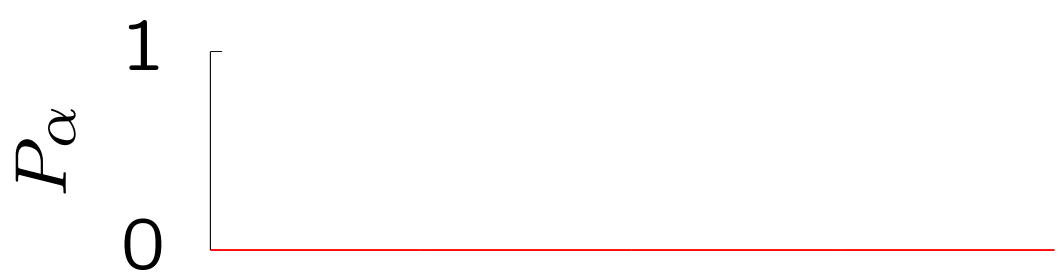
$|\Psi\rangle = \frac{1}{\sqrt{2}}|\alpha\rangle + \frac{1}{\sqrt{2}}|\beta\rangle$



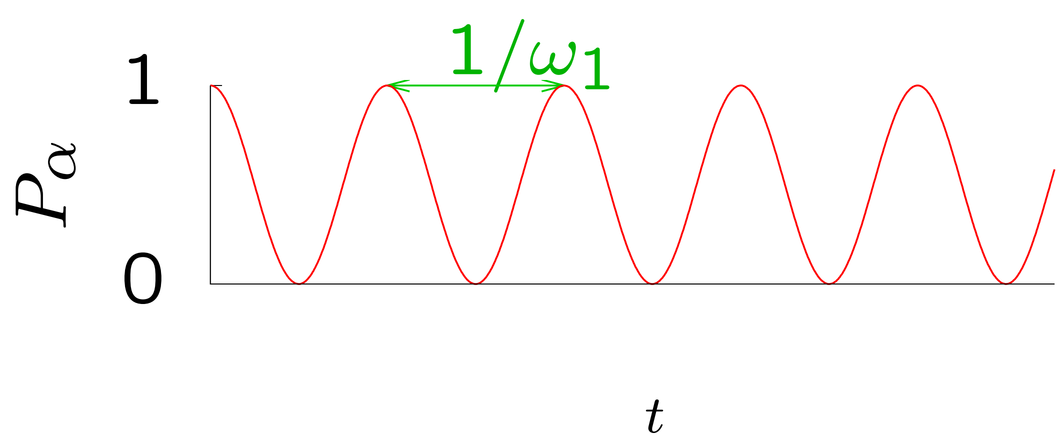
$$|\Psi\rangle(t=0) = |\alpha\rangle; \quad \hat{H} = -\gamma B_0 \hat{I}_z = \omega_0 \hat{I}_z$$



$$|\Psi\rangle(t=0) = |\beta\rangle; \quad \hat{H} = -\gamma B_0 \hat{I}_z = \omega_0 \hat{I}_z$$



$$|\Psi\rangle(t=0) = |\alpha\rangle; \quad \hat{H} = -\gamma B_1 \hat{I}_x = \omega_1 \hat{I}_x$$



- The states described by basis functions which are eigenfunctions of the Hamiltonian do not evolve (are stationary).

- It makes sense to draw *energy level diagram* for such states, with energy of each state given by the corresponding eigenvalue of the Hamiltonian.

- Energy of the $|\alpha\rangle$ state is $-\hbar\omega_0/2$ and energy of the $|\beta\rangle$ state is $+\hbar\omega_0/2$.

- The measurable quantity is the energy difference $\hbar\omega_0$, corresponding to the angular frequency ω_0.

- The states described by basis functions different from eigenfunctions of the Hamiltonian are not stationary

- They oscillate between $|\alpha\rangle$ and $|\beta\rangle$ with the angular frequency ω_1, given by the difference of the eigenvalues of the Hamiltonian ($-\hbar\omega_1/2$ and $\hbar\omega_1/2$).

- It should be stressed that eigenstates of individual magnetic moments are not eigenstates of the macroscopic ensembles of nuclear magnetic moments.

- Eigenstates of individual magnetic moments do not determine the possible result of measurement of bulk magnetization.