Non-linear waves and solitons

Lecture notes[1]

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Chapter 1

Wave equations

The existence of waves is one of the most universal phenomena in physics. From mechanical vibrations and sound to water waves to light and electromagnetic waves. Even at the most fundamental level string theory suggests that our world is governed by waves on tiny strings. In these lectures we will confine ourselves to waves in one spatial dimension, which we will see is enough to see some very interesting and rich phenomena.

The waves you have encountered so far are solutions of what is often referred to simply as **the wave equation**. In one dimension it takes the form

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = 0\,. \tag{1.1}$$

Here $u(x,t)$ describes the profile of the wave as a function of space (x) and time (t) and v is a constant – the velocity of the wave. This is the first wave equation to be systematically studied, starting with d'Alembert in 1746. The general solution takes the form of the sum of a right-moving and a left-moving wave[1]

$$u(x,t) = f(x - vt) + g(x + vt)\,, \tag{1.2}$$

where f and g are arbitrary functions of one variable specifying the profile of the right-moving and left-moving wave respectively at $t = 0$. Note that the sum (or **superposition**) of any two such solutions gives a new solution, this is due to the fact that the wave equation (1.1) is a **linear** equation, i.e. u enters linearly. Note also that the right-moving and left-moving wave retain their shape as a function of time and do not interact with each other. This is again a consequence of the linearity of the wave equation.

1.1 Dispersion, dissipation and non-linearity

In many situations is physics one encounters waves that do not quite satisfy this simple idealized wave equation. They may for example display the phenomena of **dispersion**, i.e.

[1] To see this notice that the wave equation factorizes as

$$0 = \left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) u = -v^2 \frac{\partial}{\partial x^-}\frac{\partial}{\partial x^+} u\,,$$

where we have defined $x^{\pm} = x \pm vt$. This implies that $\frac{\partial u}{\partial x^+} = g'(x^+)$ and therefore $u(x^-, x^+) = f(x^-) + g(x^+)$ for some functions f and g.

the wave tend to spread out with time, or **dissipation**, i.e. the amplitude decays with time and the wave loses energy to the surroundings. These two phenomena can also be modeled with simple linear wave equations. To see this we focus on the right-moving solution of the wave equation (1.1), $u(x,t) = f(x - vt)$. A wave equation which describes this is simply

$$\dot{u} + vu' = 0\,, \tag{1.3}$$

where we have defined the short-hand notation $\dot{u} = \frac{\partial u}{\partial t}$ and $u' = \frac{\partial u}{\partial x}$. If we look for a solution of the form $u = Ae^{i(\omega t - kx)}$ (we can always take the real part at the end) we find that the frequency ω and wave-number k are related to the velocity v as

$$v = \frac{\omega}{k}\,. \tag{1.4}$$

Now consider instead the wave equation

$$\dot{u} + au' + bu''' = 0\,, \tag{1.5}$$

with a, b constants. Taking again the same ansatz for u we now find

$$\omega = ak - bk^3\,. \tag{1.6}$$

The relation between ω and k is known as the **dispersion relation**. The **phase velocity** of the wave is given by ω/k and we see that for non-zero b it depends on the wave number k.[2] This means that if we superpose waves with different wave number (note that this is allowed since the equation is still linear) these will move with different velocity and spread out or disperse. Therefore adding the term bu''' has introduced dispersion. Adding instead a term cu'' one would find that ω gets an imaginary piece and therefore the solution $u = Ae^{i(\omega t - kx)}$ will have an exponentially decaying (or increasing) amplitude and therefore the wave is loosing energy to its surroundings – there is dissipation. A slight generalization of this argument shows that terms with an odd number of spatial derivatives introduce dispersion while terms with an even number of derivatives introduce dissipation.

So far we have considered linear wave equations which satisfy the superposition principle. A more realistic wave equation may also contain non-linearities, for example our simplest wave equation considered above (with $v = 1$ for simplicity) could be an approximation, valid for small wave amplitude, to the non-linear wave equation

$$\dot{u} + (1 + u)u' = 0\,. \tag{1.7}$$

Clearly the quadratic term in u means that the sum of two solutions will in general not be a solution – there is no superposition principle in this case. Such non-linear wave equations are much harder to solve, in fact in general they can only be solved numerically.[3] However it turns out that the equation above is an approximation to a more interesting equation which **can** be solved exactly. Let us first note that shifting $u \to u - 1$ the equation becomes $\dot{u} + uu' = 0$ and the coefficient of the second term can be changed to any non-zero number by rescaling

[2] The **group velocity**, the speed at which physical information in the wave travels, is given instead by $\partial\omega/\partial k$ but this does not affect the discussion here.

[3] However, in this simple case it is not hard to see that the general solution is $u(x,t) = f(x - (1 + u)t)$ for some function f and one can then in principle solve for u given the initial profile $u(x,0) = f(x)$.

u. The equation we will spend a large fraction of this course studying is this equation with the incorporation of dispersion[4]

$$\boxed{\textbf{KdV equation:} \qquad \dot{u} - 6uu' + u''' = 0} \tag{1.8}$$

and it is known as the **Korteweg-de Vries** (or KdV) equation. It is the simplest wave equation incorporating both non-linearity and dispersion. Note that the coefficients in front of each term are purely conventional and can be set to any non-zero value by rescaling u, x and t. This equation has the very special property of being '**exactly integrable**' which means that we will be able to solve it (in a sense that will become clear later on). Later on in this course we will meet a few other examples of exactly integrable equations.

1.2 The solitary wave

The story of the KdV equation goes back all the way to 1834. John Scott Russell, a Scottish engineer and naval architect, was performing experiments in water flow in channels to determine the most efficient design for canal boats when he witnessed a very interesting phenomenon. In his 'Report on Waves' from 1844 he recalls the event as follows:

> *I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel.*

He continued to study the properties of these waves, which eventually became known as **solitary waves**, in water tank experiments showing for example that their velocity is related to the (undisturbed) water depth h and amplitude a by the equation

$$v^2 = g(h + a)\,, \tag{1.9}$$

where g is the acceleration of gravity. However a theoretical understanding had to await the works of Boussinesq (1871) and Rayleigh (1876) who were able to derive Russell's empirical velocity formula. Finally Korteweg and de Vries (1895) derived the KdV equation[5] (1.8) for water waves in a shallow channel and showed that it had a solution with the properties of Russell's solitary wave.

Let us look for a solution of the KdV equation in the form of a traveling wave. To do this we take the ansatz $u(x,t) = f(x - vt)$, with f and function of one variable and v a constant to be determined. The KdV equation then reduces to

$$-vf' - 6ff' + f''' = 0\,, \tag{1.10}$$

[4] The corresponding equation with dissipation, i.e. $\dot{u} - 6uu' + u'' = 0$, is known as the **Burgers** equation and can also be solved exactly.

[5] In fact this equation had already been written down by Boussinesq 20 years earlier.

which integrates to $-vf - 3f^2 + f'' = a$ for some constant a and multiplying by f' and integrating again we find

$$-\frac{v}{2}f^2 - f^3 + \frac{1}{2}f'^2 = af + b\,, \tag{1.11}$$

with a, b integration constants. Requiring $f, f','f'' \to 0$ as $|x| \to \infty$ sets $a = b = 0$ and we have

$$f'^2 = f^2(2f + v)\,, \tag{1.12}$$

and therefore a real solution exists only if $2f + v \geq 0$. Integrating this equation gives

$$\int \frac{df}{f\sqrt{2f+v}} = \pm \int dx^-\,, \tag{1.13}$$

and the substitution $f = -\frac{1}{2}v\,\mathrm{sech}^2\, y$ (recall that $\mathrm{sech}\, x = 1/\cosh x$) gives $y = \frac{1}{2}v^{1/2}(x^- - x_0^-) = \frac{1}{2}v^{1/2}(x - vt - x_0)$ (the sign of y is clearly irrelevant) and therefore

$$u(x,t) = f(x - vt) = -\tfrac{1}{2}v\,\mathrm{sech}^2\left[\tfrac{1}{2}v^{1/2}(x - vt - x_0)\right]. \tag{1.14}$$

This is the solitary wave solution. The minus sign reflects our choice of signs in (1.8) and we see that for u to describe the amplitude of a water wave we should send $u \to -u$. Note that the greater the amplitude of the wave the narrower it is and the faster it travels. Note that the amplitude (and position x_0 at $t = 0$) uniquely specifies the solution. Notice also that from the dispersive term in the KdV equation one might have expected a wave like this to spread out in time and lose its character, but the non-linearity of the equation is perfectly balancing this tendency for the wave to disperse.

1.3 Solitons

After the paper of Korteweg and de Vries there was not much work on the subject until the 1960's. In 1965 Zabrusky and Kruskal did some computer simulations of the KdV equation and made some very interesting observations. They found that there are solutions which consist of several separated sech^2 waves, which by itself is not so surprising, but since these waves travel at different velocity they will typically collide with each other. The remarkable thing they observed is that after such a collision the waves would continue **undeformed**. Since these waves, while interacting strongly with each other, nevertheless retained their individual identity in collisions they called them solitons (cf. electron, proton, etc.) to emphasize their similarity to particles.

The fact that these sech^2 waves, or KdV solitons, can pass through each other undeformed sounds like there is a principle of superposition at work. However, it is easy to see that this not quite what is happening. While the waves retain their shape when they pass through each other they do interact. Amazingly the only effect of the interaction is to displace the waves relative to where they would have been if they had passed through each other without interacting.

Such solitons have since been found in other (typically one-dimensional) systems and are often associated with the system being integrable (solvable). We will meet a few examples later in this course.

1.4 Applications of the KdV equation

We have seen that the KdV equation is the simplest equation one can write which incorporates both dispersion and non-linearity. This also suggests that it should have many applications. Indeed, for a linear wave with dispersion in one dimension the dispersion relation takes the form

$$\omega(k) = kc(k^2)\,, \tag{1.15}$$

since only odd derivatives of u are allowed. In the long wavelength limit, $k \ll 1$, we find

$$\frac{\omega}{k} = c_0 - c_1 k^2\,. \tag{1.16}$$

This approximate dispersion relation is obtained from the wave-equation

$$\dot{u} + c_0 u' + c_1 u''' = 0\,. \tag{1.17}$$

If the wave is propagating in a continuous material then $\dot{u}$ is replaced by the 'material derivative' $\dot{u} + uu'$ where we assume the underlying fluid velocity to be proportional to u itself. In that case the equation becomes

$$\dot{u} + c_0 u' + auu' + c_1 u''' = 0\,, \tag{1.18}$$

which is transformed to the KdV equation (1.8) by the change of variables $x \to x - c_0 t$, $t \to at$. This shows that the KdV equation can be expected to describe the propagation of long wavelength waves in various circumstances.

Chapter 2

Elementary solutions of the KdV equation

In the last chapter we derived the solitary wave solution of the KdV equation (1.8). Assuming a solution of the form of a right-moving wave of a fixed form

$$u(x,t) = f(\xi)\,, \qquad \xi = x - vt\,. \tag{2.1}$$

With this ansatz we found that we could integrate the KdV equation twice to obtain (1.11)

$$-\frac{v}{2}f^2 - f^3 + \frac{1}{2}f'^2 = af + b\,, \tag{2.2}$$

where a, b are integration constants. We then assumed a solution which is constant far away, $f, f', f'' \to 0$ as $\xi \pm \infty$, which sets $a = b = 0$ and leads to the solitary wave solution (1.14).

2.1 General wave solutions of fixed shape

We will now derive the most general wave solutions of fixed shape. We therefore want to solve the equation

$$-\frac{v}{2}f^2 - f^3 + \frac{1}{2}f'^2 = af + b\,, \tag{2.3}$$

without assuming anything about the constants a, b. We first rearrange this equation to read

$$\frac{1}{2}f'^2 = F(f)\,, \tag{2.4}$$

where we have introduced the function

$$F(f) = f^3 + \frac{v}{2}f^2 + af + b\,. \tag{2.5}$$

We will be interested only in real bounded solutions of this equation. Clearly real solutions exist only for $F > 0$. We see from the equation that f changes monotonically until f' vanishes, which happens at a zero of F. To begin with we therefore analyze what happens near a zero of $F(f)$. Since $F(f)$ is a cubic polynomial it can have a simple, double or triple zero.

(i) Simple zero: Taylor expanding around the point $f = f_1$ where F vanishes we find

$$(f')^2 = 2(f - f_1)F'(f_1) + \mathcal{O}((f - f_1)^2)\,, \tag{2.6}$$

with solution

$$f = f_1 + \frac{1}{2}(\xi - \xi_1)^2 F'(f_1) + \mathcal{O}((\xi - \xi_1)^3)\,. \tag{2.7}$$

We see that f has a local minimum(maximum) at ξ_1 for $F'(f_1) > 0$ ($F'(f_1) < 0$).

(ii) Double zero: Taylor expanding around this point we find

$$(f')^2 = (f - f_1)^2 F''(f_1) + \mathcal{O}((f - f_1)^3)\,. \tag{2.8}$$

We see that we must have $F''(f_1) \geq 0$ and the solution is

$$f - f_1 = ce^{\pm\xi\sqrt{F''(f_1)}}\,, \qquad \text{as} \quad \xi \to \mp\infty\,, \tag{2.9}$$

with c some constant, for the solution to be bounded. The solution extends from $-\infty$ to $+\infty$, where $f \to f_1$, with a bump somewhere in between.

(iii) Triple zero: Here $F(f) = (f - f_1)^3$ so that

$$\frac{1}{2}(f')^2 = (f - f_1)^3\,, \tag{2.10}$$

with solution

$$f = f_1 + \frac{2}{(\xi \pm c)^2}\,, \tag{2.11}$$

for some constant c. This solution is however unbounded at $\xi = c$ and we will therefore discard it.

We conclude that there are only two possibilities: Either f' changes sign across f_1 (since that point is a local minimum or maximum) or $f' \to 0$ as $\xi \to \pm\infty$.

Consider first the situation where F crosses zero from below at a simple zero and remains positive to the right of the crossing (so the solution exists there). If $f'(\xi_0) > 0$ for some ξ_0 then, since f is increasing, $F(f) > 0$ for all $\xi > \xi_0$. But that means that $f \to +\infty$ as $\xi \to +\infty$ and the solution is unbounded. If instead $f'(\xi) < 0$ f will decrease until it reaches f_1. This is a simple zero so f' changes sign and once again $f \to +\infty$ as $\xi \to +\infty$ and the solution is unbounded.

This leaves only two possible cases to analyze. Either F has a simple zero at f_1, where it crosses from negative to positive, and a double zero at f_2 where it touches the line $F = 0$ and then is positive again. Or F has a simple zero at f_1, where it crosses from negative to positive, a simple zero at f_2, where it crosses from positive to negative, and finally a simple zero at f_3 where it again crosses from negative to positive. In the first case the solution has a minimum at $f = f_3$ since $F(f_3) > 0$ and approaches $f = f_1$ as $\xi \pm \infty$. This is the solitary wave solution we found previously.

It therefore remains only to analyze the case with three simple zeros. The solution can exist between $f = f_2$ and $f = f_3$ since F is positive there. It has a local maximum at f_2 ($F'(f_2) < 0$) and a local minimum at f_3 ($F'(f_3) > 0$). At these points f' changes sign.

Therefore the solution oscillates between f_2 and f_3. Half the period of the solution is given by the distance from the minimum to the maximum

$$\int_{\xi_3}^{\xi_2} d\xi = \int_{f_3}^{f_2} \frac{df}{f'} = \int_{f_3}^{f_2} \frac{df}{\sqrt{2F(f)}} . \tag{2.12}$$

The solution f itself is implicitly given by the integral

$$\xi = \xi_3 \pm \int_{f_3}^{f} \frac{dg}{\sqrt{2F(g)}} , \tag{2.13}$$

with the $\pm$ according to whether $f' > 0$ or $f' < 0$. Since $F(g)$ is cubic with simple roots are f_1, f_2, f_3 we can write

$$F(g) = (g - f_1)(g - f_2)(g - f_3) , \qquad f_1 > f_2 > f_3 . \tag{2.14}$$

And substituting

$$g = f_3 + (f_2 - f_3)\sin^2\theta \tag{2.15}$$

in the integral we find

$$\xi = \xi_3 \pm \sqrt{\frac{2}{f_1 - f_3}} \int_0^{\phi} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}} , \qquad m = \frac{f_2 - f_3}{f_1 - f_3} \leq 1 , \tag{2.16}$$

with

$$f = f_3 + (f_2 - f_3)\sin^2\phi . \tag{2.17}$$

Consider the function

$$v = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}} , \qquad 0 \leq m \leq 1 . \tag{2.18}$$

Let us compare it to the function

$$w = \int_0^{\psi} \frac{dt}{\sqrt{1 - t^2}} . \tag{2.19}$$

Making the substitution $t = \sin\theta$ we find that $w \arcsin\psi$ or $\sin w = \psi$.

In analogy with this Jacobi defined two new inverse functions as

$$\operatorname{sn} v = \sin\phi , \qquad \operatorname{cn} v = \cos\phi . \tag{2.20}$$

They are two of the **Jacobi elliptic functions**. They are often written

$$\operatorname{sn}(v|m) , \qquad \operatorname{cn}(v|m) , \tag{2.21}$$

denoting also the dependence on the parameter m. In the special cases $m = 0$ and $m = 1$ these elliptic functions reduce to more familiar functions. Setting $m = 0$ we find $v = \phi$ so that

$$\operatorname{sn}(v|0) = \sin\phi = \sin v , \qquad \operatorname{cn}(v|0) = \cos\phi = \cos v . \tag{2.22}$$

In this sense the functions sn and cn are a generalization of sin and cos, as suggested by their names. Setting $m = 1$ we also find an integral we can do and one finds for example

$$\operatorname{cn}(v|1) = \cos\phi = \frac{1}{\cosh v} = \operatorname{sech} v , \tag{2.23}$$

precisely the function that appeared in the solitary wave solution. Note that cn and sn are periodic functions except for at $m = 1$ where the periodicity is lost.

The solution we found above to the KdV equation (2.16) and (2.17) can now be expressed using these elliptic functions. Since

$$f = f_3 + (f_2 - f_3)\sin^2\phi = f_2 - (f_2 - f_3)\cos^2\phi\,, \tag{2.24}$$

we find that the solution takes the form

$$f = f_2 - (f_2 - f_3)\,\mathrm{cn}^2\left(\sqrt{\frac{f_1 - f_3}{2}}\,(\xi - \xi_3)\middle|\, m\right)\,, \qquad m = \frac{f_2 - f_3}{f_1 - f_3}\,. \tag{2.25}$$

This solution was found by Korteweg and de Vries in their 1895 paper and they dubbed the solution the **cnoidal wave**. The solution depends on three parameters $f_1 \geq f_2 \geq f_3$, the roots $F(f)$ defined in (2.5). From this equation we see that the speed of the wave is given by $v = -2(f_1 + f_2 + f_3)$.

2.2 Limits of the cnoidal wave

The cn function has two limits were it reduces to more familiar function, when the parameter takes the limiting values $m = 0$ and $m = 1$. Correspondingly there are two limits in which the cnoidal wave solution simplifies. Consider first the limit where the amplitude of the wave, defined as $a = \frac{1}{2}(f_2 - f_3)$, tends to zero. Since $m = 2a/(f_1 - f_3)$ we find that $m \to 0$ as $a \to 0$ so that $\mathrm{cn}(v|m) \to \cos v$ and cnoidal wave solution (2.25) becomes

$$f \sim f_2 - 2a\cos^2\left(\sqrt{\frac{f_1 - f_3}{2}}\,(\xi - \xi_3)\right) \qquad \text{as} \qquad a \to 0\,. \tag{2.26}$$

Setting $k = \sqrt{2(f_1 - f_3)}$ and $\hat{f}_2 = f_2 - a$ this can be written as

$$f = \hat{f}_2 - a\cos\left(k(x - vt - x_0)\right) + \mathcal{O}(a^2)\,. \tag{2.27}$$

This is a linear wave of amplitude a oscillating about the level $f = \hat{f}_2$. Furthermore we have

$$\omega = kv = -2k(f_1 + 2\hat{f}_2) = -k(k^2 + 6\hat{f}_2) + \mathcal{O}(a)\,. \tag{2.28}$$

When $a \to 0$ this reduces to the dispersion relation for the linear equation

$$\dot{u} - 6\hat{f}_2 u' + u''' = 0\,, \tag{2.29}$$

obtained by linearizing the KdV equation setting $u \to u + \hat{f}_2$ and letting $|u| \ll 1$. Therefore the limit of small amplitude of the cnoidal wave gives a linear wave with the correct dispersion relation.

The solitary wave limit requires the two roots f_1, f_2 to become a double root. We therefore take the limit $f_2 \to f_1^-$ with f_3 fixed. This implies

$$m = \frac{f_2 - f_3}{f_1 - f_3} \to 1^- \tag{2.30}$$

so that $\mathrm{cn}(v|m) \to \operatorname{sech} v$ and the cnoidal wave solution (2.25) reduces to

$$f \to f_1 - (f_1 - f_3)\operatorname{sech}^2\left(\sqrt{\frac{f_1 - f_3}{2}}(x - vt - x_0)\right) . \tag{2.31}$$

Setting $f_1 - f_3 = \frac{1}{2}a$ this becomes

$$f \to f_1 - \frac{a}{2}\operatorname{sech}^2\left(\frac{\sqrt{a}}{2}(x + 6f_1t - at - x_0)\right) . \tag{2.32}$$

This is the original solitary wave transformed by $u \to u + f_1$ and $(x,t) \to (x + 6f_1t, t)$. Alternatively we may set f_1 and note that this coindices with (1.14) (setting $v = a$).

2.3 Other solutions of the KdV equation

The KdV equation has many more solutions than the ones we've met so far. Examples of simple type are so-called *similarity solutions* and *rational solutions.* Similarity solutions are for examples solutions of the form

$$u(x,t) = t^m f(\eta) , \qquad \eta = xt^n , \tag{2.33}$$

with the integers m, n chosen so that f satisfies an ordinary differential equation. For example the non-linear equation

$$\dot{u} + uu' = 0 , \tag{2.34}$$

has a solution

$$u(x,t) = t^m f(xt^n) \tag{2.35}$$

if $m + n = -1$ and

$$mf - (1+m)\eta f' + ff' = 0 . \tag{2.36}$$

Setting $m = 0$ we find that either $f' = 0$, so that f =constant, or $f = \eta$. Thus $u(x,t) = \frac{x}{t}$ is a similarity solution of (2.34).

Similarly the dispersive equation

$$\dot{u} + u''' = 0 , \tag{2.37}$$

has a solution $u(x,t) = f(xt^{-1/3})$ with

$$-\frac{1}{3}\eta f' + f''' = 0 , \tag{2.38}$$

which can be solved in terms of Airy functions.

Combining these two examples we have the KdV equation

$$\dot{u} - 6uu' + u''' = 0 , \tag{2.39}$$

which can be solved by setting

$$u(x,t) = -(3t)^{-2/3} f(\eta) , \qquad \eta = x(3t)^{-1/3} \tag{2.40}$$

with

$$f''' + (6f - \eta)f' - 2f = 0\,. \tag{2.41}$$

This equation can be reduced to a so-called Painlevé equation, with solution describing a wave profile that decays as $\eta \to +\infty$ and oscillates as $\eta \to -\infty$. The appearance of an equation of Painlevé type is directly related to the existence of soliton solutions.

Finally we have the solutions of rational type, i.e. solutions which are rational functions of the variables. We have already met one example, the solution $u(x,t) = \frac{x}{t}$ of (2.34) which is also a similarity solution. The KdV equation

$$\dot{u} - 6uu' + u''' = 0\,, \tag{2.42}$$

also has a simple rational solution. Let's assume that

$$u(x,t) = u(x)\,, \qquad u, u', u'' \to 0 \quad \text{as} \quad |x| \to \infty\,. \tag{2.43}$$

The the equation reduces to

$$-6uu' + u''' = 0\,, \tag{2.44}$$

which is not hard to solve yielding $u = 2/x^2$ (picking the pole to be at $x = 0$). This is essentially the solution corresponding to $F(f)$ in (2.5) having a triple zero encountered before.

In fact there exists a whole *hierarchy* of rational solutions of the KdV equation. The next one is

$$u(x,t) = \frac{6x(x^3 - 24t)}{(x^3 + 12t)^2}\,. \tag{2.45}$$

In the KdV case these solutions are all singular, but there are other integrable equations for which this is not the case.

Chapter 3

The scattering and inverse scattering problems

So far we have discussed some special solutions of the KdV equation

$$\dot{u} - 6uu' + u''' = 0\,, \tag{3.1}$$

such as traveling waves of fixed form. Our next task is to understand how to solve the general initial value problem for (3.1). That is to find $u(x,t)$ for all $t > 0$ given the initial condition $u(x,0)$. This problem turns out to be related to the scattering problem of quantum mechanics. We therefore need to first recall how that works.

The scattering problem in quantum mechanics involves solving the time independent Schrödinger equation (in 1 dimension)

$$\psi'' + (\lambda - u)\psi = 0\,, \tag{3.2}$$

where $u(x)$ is the potential and λ is the eigenvalue, i.e. the energy. The *scattering problem* is to determine the eigenvalues λ given a potential $u(x)$. Mathematically this is called a *Sturm-Liouville* problem. Less familiar is the fact that the scattering data, i.e. the form of $\psi(x;\lambda)$ as $x \to \pm\infty$, can in fact determine uniquely the potential $u(x)$ which gave rise to these data. This is the *inverse scattering problem.*

The fact that we call the potential u by the same letter as a solution to the KdV equation is not an accident. The relation between $u(x)$ and the KdV solution $u(x,t)$ will become clear later.

3.1 The scattering problem

For appropriate solutions of

$$\psi'' + (\lambda - u)\psi = 0\,, \tag{3.3}$$

to exist we will require that $u(x)$ decays fast enough at infinity. In fact we need that (Faddeev 1958)

$$\int_{-\infty}^{\infty} (1 + |x|)|u(x)|dx < \infty\,. \tag{3.4}$$

The reasons for this condition are technical and won't be discussed here. We will also typically assume that $u(x)$ is infinitely differentiable, although this condition is not necessary. A familiar fact about this eigenvalue problem from quantum mechanics is that there are two types of eigenvalues: those with $\lambda > 0$ (positive energy) and those with $\lambda < 0$ (negative energy). Since $u \to 0$ as $x \to \pm\infty$ we have

$$\psi'' \sim -\lambda\psi\,. \tag{3.5}$$

So for $\lambda > 0$ the solution is

$$\lambda > 0: \qquad \psi \sim ae^{i\sqrt{\lambda}x} + be^{-i\sqrt{\lambda}x} \qquad \text{as} \qquad x \to \pm\infty\,. \tag{3.6}$$

While for $\lambda < 0$ we have

$$\lambda < 0: \qquad \psi \sim ce^{-\sqrt{-\lambda}x} + de^{\sqrt{-\lambda}x} \qquad \text{as} \qquad x \to \pm\infty\,. \tag{3.7}$$

This solution is unbounded as $x \to +\infty$ unless we set $d = 0$. This condition leads to a discrete spectrum of allowed eigenvalues λ (energies). Physically these solutions correspond to bound states since the wave function decays at infinity. Whereas for $\lambda > 0$ we need no extra condition and we have a continuous spectrum. These solutions are called scattering solutions since they oscillate at infinity rather than decay. To summarize we have

$\lambda < 0$	Neg. Energy	Bound state solutions	Decay at infinity	Discrete spectrum
$\lambda > 0$	Pos. Energy	Scattering solutions	Oscillate at infinity	Continuous spectrum

For some $u(x)$ there may not exist a discrete spectrum at all (e.g. $u(x) \geq 0$).

Integrating the Scrödinger equation twice one finds that ψ is continuous. So we consider continuous, bounded, and usually at least once differentiable eigenfunctions ψ.

For the discrete eigenvalues the eigenfunctions decay exponentially at $x \to \pm\infty$ so they can be integrated

$$\int_{-\infty}^{\infty} |\psi| dx < \infty \qquad \text{and} \qquad \int_{-\infty}^{\infty} |\psi|^2 dx = \int_{-\infty}^{\infty} \psi^2 dx < \infty\,. \tag{3.8}$$

The last equality says that ψ is *square integrable* (recall that ψ is real for the discrete eigenvalues). This is not true for the continuous eigenfunctions (in that case $\int_{-\infty}^{\infty} \psi^2 dx = \infty$).

We introduce a convenient representation for the eigenvalues and eigenfunctions as follows. For the discrete spectrum we write

$$\kappa_n = \sqrt{-\lambda_n} \qquad n = 1, 2, 3, \ldots, N\,, \tag{3.9}$$

and we order them according to $\kappa_1 < \kappa_2 < \cdots < \kappa_N$ (we won't consider cases with degenerate eigenvalues). Then

$$\psi_n(x) \sim c_n e^{-\kappa_n x} \qquad \text{as} \qquad x \to +\infty\,, \tag{3.10}$$

where the constant c_n is fixed by normalizing so that $\int_{-\infty}^{\infty} \psi^2 dx = 1$.

For the continous spectrum we write

$$k = \sqrt{\lambda} \tag{3.11}$$

and we define the solution with the following oscillatory behavior at infinity

$$\hat{\psi}(x;k) = \begin{cases} e^{-ikx} + be^{ikx} & \text{as} \qquad x \to +\infty \\ ae^{-ikx} & \text{as} \qquad x \to -\infty \end{cases} \tag{3.12}$$

Physically this corresponds to an incident wave of unit amplitude coming in from $x = +\infty$, a transmitted wave of amplitude a going out to $x = -\infty$ and a reflected wave of amplitude b going back out to $x = +\infty$. The complex constants $a(k)$ and $b(k)$ can be determined uniquely for a given $u(x)$.

Consider two different discrete eigenfunctions

$$\psi_n'' + (\kappa_n^2 - u)\psi_n = 0 \qquad \psi_m'' + (\kappa_m^2 - u)\psi_m = 0\,. \tag{3.13}$$

We find

$$(\kappa_n^2 - \kappa_m^2)\psi_n\psi_m = \psi_n\psi_m'' - \psi_m\psi_n'' = (W(\psi_n, \psi_m))'\,, \tag{3.14}$$

where the *Wronskian* is defined as $W(\alpha, \beta) = \alpha\beta' - \beta\alpha'$. Integrating we find

$$(\kappa_n^2 - \kappa_m^2)\int_{-\infty}^{\infty} \psi_n\psi_m dx = [W(\psi_n, \psi_m)]_{-\infty}^{\infty}\,. \tag{3.15}$$

Since $\psi \to 0$ as $x \to \pm\infty$ the RHS vanishes as we find

$$\int_{-\infty}^{\infty} \psi_n\psi_m dx = 0 \qquad \text{for} \qquad m \neq n\,, \tag{3.16}$$

i.e. the discrete eigenfunctions are orthogonal. The continous eigenfunctions $\hat{\psi}$ are also orthogonal to all the discrete eigenfunctions. (In fact the discrete and continuous eigenfunctions form a *complete set*, so that any square integrable function can be represented as a linear combination of ψ_ns and an integral of $\hat{\psi}$ over k.)

If θ, ϕ are two solutions with the same eigenvalue we have

$$(W(\theta, \phi))' = 0 \qquad \Rightarrow \qquad W(\theta, \phi) = const. \tag{3.17}$$

If ϕ is proportional to θ then the Wronskian vanishes. For the continuous eigenfunction $\hat{\psi}$ it is easy to see that its complex conjugate $\hat{\psi}^*$ has the same eigenvalue. We can find the constant Wronskian by evaluating it at $x = \pm\infty$ giving

$$W(\hat{\psi}, \hat{\psi}^*) = 2ik|a|^2 \tag{3.18}$$

and

$$W(\hat{\psi}, \hat{\psi}^*) = 2ik(1 - |b|^2) \tag{3.19}$$

respectively. These expressions must be equal and we find the condition

$$|a|^2 + |b|^2 = 1\,. \tag{3.20}$$

In the scattering problem a, b are transmission and reflection coefficients and this condition follows from energy conservation.

To learn more about the solution of the scattering problem we have to consider specific forms of the potential $u(x)$.

Example 1: delta function

We take

$$u(x) = -U_0\delta(x) \tag{3.21}$$

with U_0 a constant and $\delta(x)$ the Dirac delta function. Integrating the Schrödinger equation from $-\epsilon$ to ϵ gives

$$[\psi']_{-\epsilon}^{\epsilon} + \int_{-\epsilon}^{\epsilon} (\lambda + U_0\delta(x))\psi(x)dx = 0 \tag{3.22}$$

so that the discontinuity of the derivative is given by

$$\mathrm{disc}(\psi') = \lim_{\epsilon\to 0}(\psi'(\epsilon) - \psi'(-\epsilon)) = -U_0\psi(0)\,. \tag{3.23}$$

We see that ψ is continuous but not differentiable at $x = 0$. The discrete eigenfunctions are

$$\psi_n(x) = \begin{cases} \alpha_n e^{-\kappa_n x} & x > 0 \\ \beta_n e^{\kappa_n x} & x < 0 \end{cases} \tag{3.24}$$

since $u = 0$ for $x \neq 0$. For ψ_n to be continuous we need $\alpha_n = \beta_n$ and normalizing we find

$$1 = \int_{-\infty}^{\infty} \psi^2 dx = \int_{-\infty}^{0} \alpha_n^2 e^{2\kappa_n x}dx + \int_{0}^{\infty} \alpha_n^2 e^{-2\kappa_n x}dx = \frac{\alpha_n^2}{\kappa_n}\,, \tag{3.25}$$

so that $\alpha_n = \sqrt{\kappa_n}$. Finally the discontinuity of ψ' requires

$$\mathrm{disc}(\psi_n') = -2\kappa_n\alpha_n = -U_0\psi_n(0) = -U_0\alpha_n\,, \tag{3.26}$$

so that $\kappa_n = \frac{1}{2}U_0$. There is only one eigenvalue $\lambda_1 = -\frac{1}{4}U_0^2$ ($\kappa_1 = \frac{1}{2}U_0$) and only if $U_0 > 0$ since $\alpha_n = \sqrt{\kappa_n} = \sqrt{U_0/2}$ must be real.

Next we consider the continuous eigenfunctions. We have

$$\hat\psi(x;k) = \begin{cases} e^{-ikx} + be^{ikx} & \text{as} \quad x > 0 \\ ae^{-ikx} & \text{as} \quad x < 0 \end{cases}\,. \tag{3.27}$$

Continuity at $x = 0$ requires $a = 1 + b$. The discontinuity of ψ' at $x = 0$ gives

$$\mathrm{disc}(\hat\psi') = 2ikb = -U_0(1 + b)\,, \tag{3.28}$$

or

$$b(k) = -\frac{U_0}{U_0 + 2ik}\,. \tag{3.29}$$

We see that in this example there is always a continuous spectrum but only a discrete spectrum for $U_0 > 0$ (negative potential) with only one bound state.

Note that the pole in $b(k)$, in the upper half complex plane, at $k = -\frac{i}{2}U_0$ corresponds to $\lambda = k^2 = -\frac{1}{4}U_0$ which is the discrete eigenvalue! This is a general result that allows the discrete eigenvalues to be determined from the continuous eigenfunctions.

Next we consider another example of a potential which we will need when we discuss the KdV equation.

Example 2: sech^2 function

We take the potential to be

$$u(x) = -U_0 \operatorname{sech}^2 x\,, \tag{3.30}$$

with U_0 a constant. The Schrödinger equation becomes

$$\psi'' + (\lambda + U_0 \operatorname{sech}^2 x)\psi = 0\,. \tag{3.31}$$

It is convenient to introduce a new variable $T = \tanh x$ $(-1 < T < 1$ for $, -\infty < x < \infty)$. We have

$$\frac{d}{dx} = \operatorname{sech}^2 x \frac{d}{dT} = (1-T^2)\frac{d}{dT} \tag{3.32}$$

and the Schrödinger equation takes the form

$$(1-T^2)\frac{d}{dT}\left[(1-T^2)\frac{d\psi}{dT}\right] + \left(\lambda + U_0(1-T^2)\right)\psi = 0\,, \tag{3.33}$$

or equivalently

$$\frac{d}{dT}\left[(1-T^2)\frac{d\psi}{dT}\right] + \left(U_0 + \frac{\lambda}{1-T^2}\right)\psi = 0\,. \tag{3.34}$$

This is in fact the *associated Legendre equation*. This is the equation you solve to find the spherical harmonics, for example when finding the wave functions for the hydrogen atom. To use our knowledge from that case we will assume, to start with, that

$$U_0 = \ell(\ell+1)\,, \tag{3.35}$$

where ℓ is an integer (the angular momentum quantum number in the case of the hydrogen atom). Consider the discrete eigenvalues

$$\lambda = -\kappa^2 < 0\,. \tag{3.36}$$

Bounded solutions exist only for $\kappa_m = m$ $(m = 1, 2, \ldots, \ell)$, as is familiar from the case of the hydrogen atom (where m is the quantum number associated to the angular momentum in the z-direction). The eigenfunctions are (up to normalization) the *associated Legendre functions* (in the spherical harmonics case we have $T = \cos\theta$)

$$P_\ell^m(T) = (-1)^m(1-T^2)^{m/2}\frac{d^m}{dT^m}P_\ell(T)\,, \tag{3.37}$$

where

$$P_\ell(T) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dT^\ell}(T^2-1)^\ell \tag{3.38}$$

is the *Legendre polynomial* of degree ℓ.

For example for $\ell = 2$ $(U_0 = 6)$ we have

$$\psi_1 \propto P_2^1(\tanh x) = -3\tanh x \operatorname{sech} x\,, \qquad \psi_2 \propto P_2^2(\tanh x) = 3\operatorname{sech}^2 x\,. \tag{3.39}$$

Normalizing we find

$$\psi_1 = \sqrt{\frac{3}{2}}\tanh x \operatorname{sech} x\,, \qquad \psi_2 = \frac{\sqrt{3}}{2}\operatorname{sech}^2 x\,, \tag{3.40}$$

corresponding to eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -4$, respectively.

Now we turn to the continuous spectrum, $\lambda = k^2$, for which the analysis is bit more involved. We will keep U_0 arbitrary in the following. Recall that we defined the eigenfunctions behaving asymptotically as

$$\hat{\psi}(x;k) = \begin{cases} e^{-ikx} + be^{ikx} & \text{as} \quad x \to +\infty \\ ae^{-ikx} & \text{as} \quad x \to -\infty \end{cases} \tag{3.41}$$

The solution of the associated Legendre equation with this behavior involves a *hypergeometric function.* The hypergeometric function $F(a,b;c;z)$ (often ${}_2F_1(a,b;c;z)$) is the solution of the second order ODE

$$z(1-z)F'' + (c-(a+b+1)z)F' - abF = 0\,, \tag{3.42}$$

where a, b, c are constants and z is the variable. Relating this equation to the associated Legendre equation gives the solution with the correct behavior at infinity

$$\hat{\psi}(x;k) = a(k)2^{ik}(\operatorname{sech} x)^{-ik} F(c_+, c_-; 1-ik; \tfrac{1}{2}(1+T))\,, \qquad c_\pm = \tfrac{1}{2} - ik \pm \sqrt{U_0 + \tfrac{1}{4}}\,. \tag{3.43}$$

As $x \to -\infty$ we have $\frac{1}{2}(1+T) \to 0$ and

$$\hat{\psi}(x;k) \sim a(k)e^{-ikx} \tag{3.44}$$

while $x \to +\infty$ corresponds to $\frac{1}{2}(1+T) \to 1$ giving

$$\hat{\psi}(x;k) \sim a(k)\frac{\Gamma(1-ik)\Gamma(ik)}{\Gamma(c_+)\Gamma(c_-)}e^{-ikx} + a(k)\frac{\Gamma(1-ik)\Gamma(-ik)}{\Gamma(1-ik-c_+)\Gamma(1-ik-c_-)}e^{ikx}\,. \tag{3.45}$$

Comparing to (3.41) we read off

$$a(k) = \frac{\Gamma(c_+)\Gamma(c_-)}{\Gamma(1-ik)\Gamma(ik)}\,, \qquad b(k) = a(k)\frac{\Gamma(1-ik)\Gamma(-ik)}{\Gamma(1-ik-c_+)\Gamma(1-ik-c_-)}\,. \tag{3.46}$$

The expression for $b(k)$ can be simplified using the identity

$$\Gamma(\tfrac{1}{2}+z)\Gamma(\tfrac{1}{2}-z) = \frac{\pi}{\cos(\pi z)}\,. \tag{3.47}$$

Using this we have

$$\Gamma(1-ik-c_+)\Gamma(1-ik-c_-) = \Gamma(\tfrac{1}{2} - \sqrt{U_0+\tfrac{1}{4}})\Gamma(\tfrac{1}{2} + \sqrt{U_0+\tfrac{1}{4}}) = \frac{\pi}{\cos\left(\pi\sqrt{U_0+\frac{1}{4}}\right)}\,, \tag{3.48}$$

so that

$$b(k) = \frac{a(k)}{\pi}\cos\left(\pi\sqrt{U_0+\tfrac{1}{4}}\right)\Gamma(1-ik)\Gamma(-ik)\,. \tag{3.49}$$

But now we notice something curious: for certain U_0 $b(k)$ vanishes for all k. Namely if

$$\sqrt{U_0+\tfrac{1}{4}} = \ell + \tfrac{1}{2} \qquad \Rightarrow \qquad U_0 = (\ell+\tfrac{1}{2}) - \tfrac{1}{4} = \ell(\ell+1)\,, \tag{3.50}$$

the same values we considered before when discussing the discrete eigenvalues. Recall that $b(k)$ is the amplitude of the reflected wave in the scattering problem. Potential for which this vanishes are called *reflectionless potentials*. The case $U_0 = \ell(\ell+1)$, with discrete eigenfunctions given by the associated Legendre functions, is therefore an example of a reflectionless potential.

Recall from last time that the poles in $a(k), b(k)$ in the upper half-plane correspond to the discrete eigenvalues. We can therefore use our expressions for $a(k), b(k)$ to learn about the discrete spectrum for general U_0.

The poles in $a(k), b(k)$ arise from poles in the Gamma functions and $\Gamma(x)$ has poles at $x = -m$ for $m = 0, 1, 2, \ldots$. Poles in the upper half-plane can come only from c_- and we find

$$-m = c_- = \tfrac{1}{2} - ik - \sqrt{U_0 + \tfrac{1}{4}} \tag{3.51}$$

or

$$k = i\left[\sqrt{U_0 + \tfrac{1}{4}} - (m + \tfrac{1}{2})\right]. \tag{3.52}$$

If $U_0 > 0$ there are a finite number of poles in the upper half-plane, i.e. a finite number of discrete eigenvalues. The eqigenfunctions are a generalization of the associated Legendre functions P_ℓ^m to non-integer ℓ, m.

In summary we have learned that for $U_0 > 0$ there are a finite number of discrete eigenvalues, while for $U_0 < 0$ there is only a continuous spectrum. We have also seen that for some special values of U_0 ($U_0 = \ell(\ell+1)$) the potential is reflectionless, i.e. $b(k) = 0$ for all k.

This completes our study of the scattering problem. We now turn to the inverse problem.

3.2 The inverse scattering problem

Eigenvalue problems such as the scattering problem were fairly well understood by about 1850, but the inverse problem was not solved until 1951. Physically the problem is like the problem of finding the shape of a drum from the sounds it makes. In our terms, for example, given $b(k)$ can we find $u(x)$? This is much less straightforward than the scattering problem. This section will therefore be a bit more technical.

To motivate the first part of the calculation we start with the classical linear wave equation

$$\ddot{\phi} - \phi'' = 0. \tag{3.53}$$

One way to solve it is to use the Fourier transform. Writing (we will set $t = z$ for later convenience)

$$\phi(x, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(x; k) e^{-ikz} dk \tag{3.54}$$

then

$$\psi(x; k) = \int_{-\infty}^{\infty} \phi(x, z) e^{ikz} dz \tag{3.55}$$

is the Fourier transform of $\phi(x, z)$ with respect to z. The wave equation becomes

$$\psi'' + k^2 \psi = 0. \tag{3.56}$$

Suppose we are interested in solutions with

$$\psi \sim e^{ikx} \qquad \text{as} \qquad x \to +\infty. \tag{3.57}$$

This can be done by taking

$$\phi(x,z) = \delta(x-z) + K(x,z)\,, \tag{3.58}$$

with $K(x,z)=0$ for $z<x$ and satisfying $\ddot{K} - K'' = 0$, since then

$$\psi(x;k) = e^{ikx} + \int_x^\infty K(x,z)e^{ikz}dz\,, \tag{3.59}$$

which has the required behavior as $x \to +\infty$.

The equation we are really interested in is

$$\psi'' + (k^2 - u)\psi = 0\,, \tag{3.60}$$

the Schrödinger equation with $\lambda = k^2$, corresponding to the continuous spectrum. If differs from (3.56) only by the $u\psi$-term. We look again for solutions with

$$\psi \sim e^{ikx} \qquad \text{as} \qquad x \to +\infty\,. \tag{3.61}$$

We may suppose that ψ is again of the form

$$\psi_+(x;k) = e^{ikx} + \int_x^\infty K(x,z)e^{ikz}dz\,, \tag{3.62}$$

The subscript $+$ denotes a solution with boundary condition at $x \to +\infty$. The original problem is now traded for the problem of finding $K(x,z)$. To find what the Schrödinger equation implies for K we first calculate the derivatives of ψ_+. We find

$$\psi_+' = ike^{ikx} - \widehat{K}e^{ikx} + \int_x^\infty K'(x,z)e^{ikz}dz \tag{3.63}$$

where we introduced $\widehat{K}(x) = K(x,x)$. The second derivative becomes

$$\psi_+'' = e^{ikx}\left(-k^2 - \widehat{K}' - ik\widehat{K} - K'|_{z=x}\right) + \int_x^\infty K''e^{ikz}dz\,. \tag{3.64}$$

Note that $\widehat{K}' = (K' + \dot{K})|_{z=x}$. It will be convenient to rewrite ψ_+ itself in a similar way. Integrating by parts twice and assuming that $K, \dot{K} \to 0$ as $z \to +\infty$ we have

$$\psi_+ = e^{ikx} + \frac{1}{ik}[Ke^{ikz}]_x^\infty - \frac{1}{ik}\int_x^\infty \dot{K}e^{ikz}dz = e^{ikx}\left(1 - \frac{\widehat{K}}{ik} - \frac{1}{k^2}\dot{K}|_{z=x}\right) - \frac{1}{k^2}\int_x^\infty \ddot{K}e^{ikz}dz\,. \tag{3.65}$$

Using these facts the Schrödinger equation becomes

$$\begin{aligned} 0 =& \psi_+'' + (k^2 - u)\psi_+ = e^{ikx}\left(-\widehat{K}' - K'|_{z=x} - \dot{K}|_{z=x}\right) + \int_x^\infty (K'' - \ddot{K})e^{ikz}dz - u\psi_+ \\ =& - e^{ikx}\left(u + 2\widehat{K}'\right) + \int_x^\infty (K'' - \ddot{K} - u(x)K)e^{ikz}dz\,, \end{aligned} \tag{3.66}$$

which implies

$$u = -2\widehat{K}'\,, \qquad \text{and} \qquad K'' - \ddot{K} - u(x)K = 0 \qquad (z > x)\,. \tag{3.67}$$

Together with the conditions $K, \dot{K} \to 0$ as $z \to +\infty$ this defines the problem for $K(x,z)$.

Given $u(x)$ $K(x,z)$ is a real valued function satisfying a wave equation (with z playing the role of time) with initial data at $z = x$ ($u = -2\widehat{K}' = -2(K' + \dot{K})|_{z=x}$) and at $z = +\infty$. It is well known that the solution exists and is unique.

So far we have just reformulated the scattering problem as the problem of finding $K(x,z)$ given $u(x)$. However, we can now try to invert the expression

$$\psi_+(x;k) = e^{ikx} + \int_x^\infty K(x,z)e^{ikz}dz\,, \tag{3.68}$$

to obtain K from ψ and then $u(x)$ from $u = -2\widehat{K}'$. This is possible because we have just argued that a unique K exists.

Before we attack this problem we have to relate the solution ψ_+ to the continuous eigenfunctions $\hat{\psi}$ introduced before. We can construct $\hat{\psi}$ from ψ_+ as

$$\hat{\psi} = \psi_+^* + b(k)\psi_+\,. \tag{3.69}$$

Indeed, we then find

$$\hat{\psi} \sim e^{-ikx} + b(k)e^{ikx} \qquad x \to +\infty\,, \tag{3.70}$$

as required. Using the expression for ψ_+ we obtain

$$\hat{\psi} = e^{-ikx} + b(k)e^{ikx} + \int_{-\infty}^\infty K(x,z)e^{-ikz}dz + b(k)\int_{-\infty}^\infty K(x,z)e^{ikz}dz\,, \tag{3.71}$$

since $K(x,z) = 0$ for $z < x$. Writing this as

$$\int_{-\infty}^\infty K(x,z)e^{-ikz}dz = \hat{\psi} - e^{-ikx} - b(k)e^{ikx} - b(k)\int_{-\infty}^\infty K(x,z)e^{ikz}dz \tag{3.72}$$

we can invert it by performing the inverse Fourier transform yielding

$$K(x,z) = \frac{1}{2\pi}\int_{-\infty}^\infty \left[\hat{\psi} - e^{-ikx} - b(k)e^{ikx} - b(k)\int_{-\infty}^\infty K(x,y)e^{iky}dy\right] e^{ikz}dk\,. \tag{3.73}$$

Let $F(x)$ be the Fourier transform of $b(k)$,

$$F(x) = \int_{-\infty}^\infty b(k)e^{ikx}dk\,. \tag{3.74}$$

We find

$$K(x,z) = \frac{1}{2\pi}\int_{-\infty}^\infty \left(\hat{\psi}(x;k) - e^{-ikx}\right) e^{ikz}dk - F(x+z) - \int_{-\infty}^\infty K(x,y)F(y+z)dy\,. \tag{3.75}$$

To evaluate the first integral we note that since we are assuming that there are no discrete eigenvalues $\hat{\psi}$ has no poles in the upper half complex k-plane. Furthermore the integrand

$$\left(\hat{\psi}(x;k)e^{ikx} - 1\right) e^{ik(z-x)} \tag{3.76}$$

decays exponentially at large $|k|$ with $\Im(k) > 0$ since $z > x$. We can therefore close the integration contour with a semicircle of radius R in the upper half plane. Since the semicircle part of the contour does not contribute as $R \to \infty$ and since there are no poles inside the contour we find, by Cauchy's theorem,

$$\int_{-\infty}^{\infty} \left(\hat{\psi}(x;k) - e^{-ikx} \right) e^{ikz} dk = 0 \,. \tag{3.77}$$

We therefore obtain

$$K(x,z) + F(x+z) + \int_x^{\infty} K(x,y)F(y+z)dy = 0 \qquad z > x \,. \tag{3.78}$$

This is known as the *Marchenko equation* for $K(x,z)$ given $F(x)$. Once this integral equation is solved, we will discuss how shortly, the potential is obtained from $u(x) = -2\widehat{K}'(x)$.

But first, to complete the analysis, we must take into account the discrete spectrum. The analysis leading to (3.75) is unchanged but now the evaluation of the integral involving $\hat{\psi}$ changes since now there are poles in the upper half plane corresponding to the discrete eigenvalues. Using the same contour as before (with R large enough to enclose all poles) Cauchy's residue theorem gives

$$\int_{-\infty}^{\infty} \left(\hat{\psi}(x;k) - e^{-ikx} \right) e^{ikz} dk = 2\pi i \sum_{n=1}^{N} R_n \,, \tag{3.79}$$

where $R_n = \text{Res}_{k \to i\kappa_n}(\hat{\psi} e^{ikz})$ is the residue on the nth pole. To find these residues we need to introduce the other solution with boundary condition at $x \to -\infty$

$$\psi_-(x;k) = e^{-ikx} + \int_{-\infty}^{x} L(x,z) e^{-ikz} dz \,, \tag{3.80}$$

where $L(x,z)$ plays the same role for this solution that $K(x,z)$ does for ψ_+. The relation between $\hat{\psi}$ and ψ_- is fixed by the behavior at $-\infty$ and we have

$$\hat{\psi} = a(k)\psi_- = \psi_+^* + b(k)\psi_+ \,, \tag{3.81}$$

so that

$$\psi_- = a^{-1}\psi_+^* + a^{-1} b \psi_+ \,. \tag{3.82}$$

These equations imply that ψ_+ and ψ_- are well defined at the poles $k = i\kappa_n$ and from their asymptotic behavior we have

$$\psi_+(x; i\kappa_n) \sim e^{-\kappa_n x} \qquad x \to +\infty \qquad \psi_-(x; i\kappa_n) \sim e^{\kappa_n x} \qquad x \to -\infty \tag{3.83}$$

But this is precisely the behavior of the discrete eigenfunction ψ_n so we may write

$$\psi_n(x) = c_n \psi_+(x; i\kappa_n) = d_n \psi_-(x; i\kappa_n) \,, \tag{3.84}$$

for some constants c_n, d_n. Next we will use these relations to derive some identities involving the Wronskian which will allow us to find the residues we are after.

Recall that the Wronskian $W(\psi_-, \psi_+)$ is a constant (since $\psi_\pm$ are solutions to the Schröding equation with the same eigenvalue) so we may evaluate it at $x \to +\infty$ and using (3.82) we find

$$W(\psi_-, \psi_+) = \psi_- \psi_+' - \psi_+ \psi_-' = 2ika^{-1} \,. \tag{3.85}$$

Taking a derivative of this equation with respect to k we find

$$W(\frac{d}{dk}\psi_-,\psi_+) + W(\psi_-,\frac{d}{dk}\psi_+) = 2ia^{-1} - 2ika'a^{-2}\,. \tag{3.86}$$

This equation can be used to show that a has only simple poles (see the problems). If we take instead the k-derivative of the original Schrödinger equation we find

$$\frac{d}{dk}\psi'' + 2k\psi + (k^2-u)\frac{d}{dk}\psi = 0\,. \tag{3.87}$$

Multiplying with ψ and using the Schrödinger equation again we find

$$\psi\frac{d}{dk}\psi'' + 2k\psi^2 - \psi''\frac{d}{dk}\psi = 0\,, \tag{3.88}$$

or

$$[W(\frac{d}{dk}\psi,\psi)]' = 2k\psi^2\,. \tag{3.89}$$

Taking $\psi=\psi_+$, $k=i\kappa_n$ and integrating from $-\infty$ to $+\infty$ we get

$$W_n^+(\frac{d}{dk}\psi_+,\psi_+) - W_n^-(\frac{d}{dk}\psi_+,\psi_+) = 2i\kappa_n\int_{-\infty}^{+\infty}\psi_+^2dx = \frac{2i\kappa_n}{c_n^2}\,, \tag{3.90}$$

where $W_n^{\pm}$ denotes the Wronskian evaluated at $x=\pm\infty$ and $k=i\kappa_n$. On the RHS we used the fact that $\psi_+=\psi_n/c_n$ and the normalization of the discrete eigenfunctions ψ_n.

Evaluating instead the identity (3.86) at $x=-\infty$ and $k=i\kappa_n$ we find

$$W_n^-(\frac{d}{dk}\psi_-,\psi_+) + W_n^-(\psi_-,\frac{d}{dk}\psi_+) = -2ik\left(\frac{a'}{a^2}\right)_n\,, \tag{3.91}$$

or

$$W_n^-(\frac{d}{dk}\psi_-,\frac{d_n}{c_n}\psi_-) + W_n^-(\frac{c_n}{d_n}\psi_+,\frac{d}{dk}\psi_+) = -2ik\left(\frac{a'}{a^2}\right)_n\,, \tag{3.92}$$

where the superscript n on the RHS denotes evaluation at $k=i\kappa_n$. Using (3.90) to eliminate the second term we find

$$W_n^+(\frac{d}{dk}\psi_+,\psi_+) - \frac{d_n^2}{c_n^2}W_n^-(\frac{d}{dk}\psi_-,\psi_-) = \frac{2i\kappa_n}{c_n^2} - 2\kappa_n\frac{d_n}{c_n}\left(\frac{a'}{a^2}\right)_n\,. \tag{3.93}$$

From the expression for ψ_+ in (3.62) we see that $\frac{d}{dk}\psi_+\to ike^{ikx}$ as $x\to\infty$, which means that the first term on the LHS vanishes. Similarly the second term on the LHS vanishes and we find

$$\left(\frac{a'}{a^2}\right)_n = \frac{i}{c_nd_n}\,. \tag{3.94}$$

We can now easily evaluate the residues of $\hat{\psi}e^{ikz}$. We have

$$\hat{\psi}(x;k)e^{ikz} = a(k)\psi_-(x;k)e^{ikz} = \frac{c_n}{d_n}a(k)\psi_+(x;k)e^{ikz}\,. \tag{3.95}$$

Now only $a(k)$ has poles so the residue at $k=i\kappa_n$ is

$$R_n = \frac{c_n}{d_n}\psi_+(x;i\kappa_n)e^{-\kappa_n z}\mathrm{Res}_{k\to i\kappa_n}a(k)\,. \tag{3.96}$$

Since $a(k)$ has only simple poles the residue is equal to

$$\operatorname{Res} a = \operatorname{Res} \frac{1}{a^{-1}} = \left(\frac{1}{(a^{-1})'}\right)_n = -\left(\frac{a^2}{a'}\right)_n = ic_n d_n \tag{3.97}$$

and we find

$$R_n = ic_n^2 \psi_+(x; i\kappa_n) e^{-\kappa_n z}\,. \tag{3.98}$$

Therefore we have

$$\int_{-\infty}^{\infty} \left(\hat{\psi}(x;k) - e^{-ikx}\right) e^{ikz} dk = -2\pi \sum_{n=1}^{N} c_n^2 e^{-\kappa_n z} \left(e^{-\kappa_n x} + \int_x^{\infty} K(x,y) e^{-\kappa_n y} dy\right)\,. \tag{3.99}$$

Using this in (3.75) we find again the Marchenko equation

$$K(x,z) + F(x+z) + \int_x^{\infty} K(x,y) F(y+z) dy = 0 \qquad z > x\,, \tag{3.100}$$

but with F now given by

$$F(x) = \sum_{n=1}^{N} c_n^2 e^{-\kappa_n x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} b(k) e^{ikx} dk\,. \tag{3.101}$$

This agrees with our previous result if there is no discrete spectrum. These two equations complete our formulation of the inverse scattering problem. Given the discrete spectrum, normalization constants c_n and reflection coefficient $b(k)$ the potential $u(x)$ can be determined from $K(x,z)$. For the final step we need to solve the Marchenko equation, which is what we now turn to.

3.3 The solution of the Marchenko equation

The Marchenko equation is a Fredholm integral equation, meaning that it can be written with constants as integration limits,

$$K(x,z) + F(x+z) + \int_{-\infty}^{\infty} K(x,y) F(y+z) dy = 0\,, \tag{3.102}$$

with $K(x,z) = 0$ for $z < x$. Note that x just plays the role of a parameter here. A direct way of solving this equation is by iteration. Define

$$K_1(x,z) = \begin{cases} -F(x+z) & z > x \\ 0 & z < x \end{cases} \tag{3.103}$$

and

$$\begin{aligned} K_2 &= -F(x+z) - \int_{-\infty}^{\infty} K_1(x,y) F(y+z) dy \\ K_3 &= -F(x+z) - \int_{-\infty}^{\infty} K_2(x,y) F(y+z) dy \end{aligned}$$

and so on. If $K_n(x,z) \to K(x,z)$ pointwise as $n \to \infty$ the existence of the solution is established. The infinite expansion so obtained is called the Neumann series.

This gives a formal solution to the Marchenko equation, but it is unlikely to lead to simple closed form expressions. If we are lucky and F is sufficiently simple it may be possible to instead find the solution directly. In the case relevant to the KdV equation it turns out that we are lucky and F takes a form which reduces the Marchenko equation to a standard problem.

Suppose that $F(x+z)$ is a *separable function*, i.e. that

$$F(x+z) = \sum_{n=1}^{N} X_n(x) Z_n(z)\,, \tag{3.104}$$

where N is finite. Then the Marchenko equation becomes

$$K(x,z) + \sum_{n=1}^{N} X_n(x) Z_n(z) + \sum_{n=1}^{N} Z_n(z) \int_{-\infty}^{\infty} K(x,y) X_n(y) dy = 0\,. \tag{3.105}$$

It is clear from this expression that the solution is of the form

$$K(x,z) = \sum_{n=1}^{N} L_n(x) Z_n(z)\,, \tag{3.106}$$

for some functions $L_n(x)$ that we need to find. Substituting this into the Marchenko equation we find a sum of terms each of which should vanish giving

$$L_n(x) + X_n(x) + \sum_{m=1}^{N} L_m(x) \int_{x}^{\infty} Z_m(y) X_n(y) dy = 0 \qquad n = 1, 2, \ldots, N\,. \tag{3.107}$$

Therefore the Marchenko equation is reduced to solving N algebraic equations for N unknowns, $L_n(x)$. The solution is now straightforward. We will come back to it in the next chapter when we discuss the case relevant to the KdV equation. Here we will instead just analyze two simple examples to illustrate in detail how the inverse scattering problem is solved.

Example 1: Reflection coefficient with one pole

We assume the reflection coefficient has the form

$$b(k) = -\frac{\beta}{\beta + ik} \tag{3.108}$$

for some constant $\beta > 0$. There is only one discrete eigenvalue corresponding to the pole in the upper half plane at $k = i\beta$, i.e. $\kappa_1 = \beta$. The remaining piece of scattering data we need is the normalization coefficient c_1. It can be found as follows. We have seen that we can write

$$\hat{\psi} = \psi_+^* + b(k)\psi_+\,, \tag{3.109}$$

while we have also seen that (3.98)

$$\mathrm{Res}_{k \to i\kappa_n}(\hat{\psi} e^{ikz}) = i c_n^2 \psi_+(x; i\kappa_n) e^{-\kappa_n z}\,. \tag{3.110}$$

Putting these two facts together and noting that ψ_+ has no poles we find

$$i\beta = \mathrm{Res}_{k\to i\kappa_n} b(k) = ic_1^2\,, \tag{3.111}$$

so that $c_1 = \sqrt{\beta}$ and the discrete eigenfunction goes as

$$\psi_1 \sim \sqrt{\beta}e^{-\beta x} \qquad x \to +\infty\,. \tag{3.112}$$

From the definition (3.101) we now find

$$F(x) = \beta e^{-\beta x} - \frac{\beta}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{\beta + ik}dk\,. \tag{3.113}$$

To evaluate the integral imagine closing the contour with a semicircle in the upper half plane. The semicircular part does not contribute provided that $x > 0$, since then the integrand decays exponentially. Cauchy's residue theorem then gives

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{\beta + ik}dk = 2\pi i \mathrm{Res}_{k\to i\beta}\frac{e^{ikx}}{\beta + ik} = 2\pi e^{-\beta x} \qquad x > 0\,. \tag{3.114}$$

If $x < 0$ we instead close the contour in the lower half plane and since there are no poles there we find

$$\int_{-\infty}^{\infty} \frac{e^{ikx}}{\beta + ik}dk = 0 \qquad x < 0\,. \tag{3.115}$$

Therefore we find

$$F(x) = \beta e^{-\beta x} H(-x)\,, \tag{3.116}$$

where the Heaviside step function is defined as

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}\,. \tag{3.117}$$

The Marchenko equation now gives $K(x, z) = 0$ for $x + z > 0$ since F vanishes there, while for $x + z < 0$ we get

$$K(x, z) + \beta e^{-\beta(x+z)} + \beta\int_x^{-z} K(x, y)e^{-\beta(y+z)}dy = 0\,, \tag{3.118}$$

since $F(y + z)$ vanishes for $y + z > 0$. One way to find the solution is to integrate by parts giving

$$K(x, z) + \beta e^{-\beta(x+z)} - K(x, -z) + K(x, x)e^{-\beta(x+z)} + \int_x^{-z} \frac{\partial}{\partial y}K(x, y)e^{-\beta(y+z)}dy = 0\,. \tag{3.119}$$

It is clear from this expression that one solution is $K(x, z) = -\beta$. By uniqueness this must be the required solution. We then have

$$K(x, z) = -\beta H(-x - z)\,. \tag{3.120}$$

The potential becomes

$$u(x) = -2[K(x, x)]' = 2\beta[H(-2x)]' = 2\beta[H(-x)]' = -2\beta\delta(x)\,. \tag{3.121}$$

This recovers our first example of the solution of the scattering problem, setting $\beta = U_0/2$.

Example 2: Zero reflection coefficient

Since $b(k) = 0$ only the discrete eigenvalues contribute to F. We suppose that we have two of them with

$$\psi_1 \sim c_1 e^{-\kappa_1 x}\,, \qquad \psi_2 \sim c_2 e^{-\kappa_2 x}\,, \qquad x \to +\infty\,. \tag{3.122}$$

We then obtain

$$F(x) = c_1^2 e^{-\kappa_1 x} + c_2^2 e^{-\kappa_2 x}\,. \tag{3.123}$$

We see that

$$F(x+z) = \sum_{n=1}^{2} c_n^2 e^{-\kappa_n x} e^{-\kappa_n z} \tag{3.124}$$

i.e. $F(x+z)$ is separable with $X_n(x) = c_n^2 e^{-\kappa_n x}$ and $Z_n(z) = e^{-\kappa_n z}$. The Marchenko equation gives (3.107)

$$L_n(x) + X_n(x) + \sum_{m=1}^{2} L_m(x) \int_x^\infty Z_m(y) X_n(y) dy = 0 \qquad n = 1, 2\,. \tag{3.125}$$

In particular

$$0 = L_1 + c_1^2 e^{-\kappa_1 x} + c_1^2 L_1 \int_x^\infty e^{-2\kappa_1 y} dy + c_1^2 L_2 \int_x^\infty e^{-(\kappa_1+\kappa_2) y} dy = L_1 + c_1^2 e^{-\kappa_1 x} + c_1^2 \sum_{m=1}^{2} \frac{L_m}{\kappa_1 + \kappa_m} e^{-(\kappa_1+\kappa_m)x} \tag{3.126}$$

and similarly for L_2 giving

$$L_n + c_n^2 e^{-\kappa_n x} + c_n^2 \sum_{m=1}^{2} \frac{L_m e^{-(\kappa_m+\kappa_n)x}}{\kappa_m + \kappa_n} = 0\,. \tag{3.127}$$

It is convenient to write this in matrix notation as

$$AL + X = 0 \tag{3.128}$$

with

$$L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}, \qquad X = \begin{pmatrix} c_1^2 e^{-\kappa_1 x} \\ c_2^2 e^{-\kappa_2 x} \end{pmatrix}, \tag{3.129}$$

and

$$A_{mn} = \delta_{mn} + c_m^2 \frac{e^{-(\kappa_m+\kappa_n)x}}{\kappa_m + \kappa_n}\,. \tag{3.130}$$

This system of equations can be easily generalized to the case of more than two eigenvalues. The solution is

$$L = -A^{-1} X \tag{3.131}$$

and then

$$K(x,x) = Z^T L\,, \qquad Z = \begin{pmatrix} e^{-\kappa_1 x} \\ e^{-\kappa_2 x} \end{pmatrix}. \tag{3.132}$$

The important simplifying observation is that

$$\frac{d}{dx} A_{mn} = -c_m^2 e^{-(\kappa_m+\kappa_n)x} = -X_m Z_n \tag{3.133}$$

so that

$$K(x,x) = Z_m L_m = -Z_m A^{-1}_{mn} X_n = A^{-1}_{mn} \frac{d}{dx} A_{nm} = \operatorname{tr}\left(A^{-1}\frac{d}{dx}A\right) = \frac{d}{dx}\operatorname{tr}\log A = \frac{d}{dx}\log\det A\,. \tag{3.134}$$

Therefore we find

$$u(x) = -2[K(x,x)]' = -2\frac{d^2}{dx^2}\log\det A\,. \tag{3.135}$$

In our example A is a 2×2 matrix and

$$\det A = \left(1 + \frac{c_1^2}{2\kappa_1}e^{-2\kappa_1}\right)\left(1 + \frac{c_2^2}{2\kappa_2}e^{-2\kappa_2}\right) - \frac{c_1^2 c_2^2}{(\kappa_1+\kappa_2)^2}e^{-2(\kappa_1+\kappa_2)x} \tag{3.136}$$

which gives the required potential.

As a check of this result let us retain just one eigenvalue by setting $c_2 = 0$. Then we find

$$u(x) = -2\frac{d^2}{dx^2}\log\left(1 + \frac{c_1^2}{2\kappa_1}e^{-2\kappa_1}\right) = -2\kappa_1^2\operatorname{sech}^2(\kappa_1 x + x_0)\,, \tag{3.137}$$

where we defined $c_1^2/2\kappa_1 = e^{-2x_0}$. By shifting and rescaling x this can be brought to the form $u(x) = -2\operatorname{sech}^2(x)$ and since $2 = \ell(\ell+1)$ with $\ell = 1$ this is precisely one of the class of reflectionless potentials we encountered when discussing the corresponding scattering problem.

Chapter 4

Initial value problem for the KdV equation

We are now ready to apply the techniques from the inverse scattering problem to the solution of the initial value problem for the KdV equation. Let us first summarize the solution to the inverse scattering problem, i.e. the problem of finding the potential $u(x)$ in the Schrödinger equation

$$\psi'' + (\lambda - u)\psi = 0\,, \tag{4.1}$$

given the scattering data, the asymptotic behavior of ψ. The data needed is summarized as

$$\text{Continuous spectrum:}\quad \lambda > 0,\ k = \sqrt{\lambda} \qquad \hat{\psi} \sim \begin{cases} e^{-ikx} + b(k)e^{ikx} & x \to +\infty \\ a(k)e^{-ikx} & x \to -\infty \end{cases}$$

$$\text{Discrete spectrum:}\quad \lambda < 0,\ \kappa_n = \sqrt{-\lambda_n} \quad \psi_n \sim c_n e^{-\kappa_n x} \qquad x \to +\infty \qquad (n = 1, \ldots, N)$$

We have seen in the last chapter that the solution to the inverse scattering problem is given by

$$u(x) = -2\frac{d}{dx}K(x,x)\,, \tag{4.2}$$

where $K(x,z)$ is the solution to the Marchenko equation

$$K(x,z) + F(x+z) + \int_x^\infty K(x,y)F(y+z)dy = 0\,, \tag{4.3}$$

where F encodes the scattering data as

$$F(x) = \sum_{n=1}^{N} c_n^2 e^{-\kappa_n x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)e^{ikx}dk\,. \tag{4.4}$$

4.1 Relation to the KdV equation

To connect to the KdV equation

$$\dot{u} - 6uu' + u''' = 0\,, \tag{4.5}$$

consider writing

$$u = \frac{\psi''}{\psi} \tag{4.6}$$

for some $\psi(x;t) \neq 0$, where t just enters as a parameter. We can write this as

$$\psi'' - u\psi = 0\,. \tag{4.7}$$

This is almost the Schrödinger equation we want. In fact, noting that the KdV equation is invariant under the *Galilean transformation*

$$u(x,t) \to \lambda + u(x + 6\lambda t, t) \tag{4.8}$$

for constant λ and that the x-dependence is unchanged we can simply replace u by $u - \lambda$ giving

$$\psi'' + (\lambda - u)\psi = 0\,, \tag{4.9}$$

the Schrödinger equation with eigenvalue λ. Therefore, if we could solve this equation for ψ we would find u from (4.6). This is however diffcult since u appears already in the equation for ψ. The way out is to interpret this as an inverse scattering problem.

Let $u(x,t)$ be the solution of

$$\dot{u} - 6uu' + u''' = 0 \tag{4.10}$$

with $u(x,0) = f(x)$ given. This is the initial value problem for the KdV equation. Now let $\psi(x;t)$ be the solution of

$$\psi'' + (\lambda - u)\psi = 0\,, \tag{4.11}$$

with the potential given by this $u(x,t)$ and $\lambda = \lambda(t)$. This is the Schrödinger equation but with everything depending on the extra parameter t. The solution of the KdV equation can be described in three steps:

1. At $t = 0$ we are given $u(x,0) = f(x)$. We can solve the usual scattering problem for this potential to find the scattering data $S(0) = \{b(k), \kappa_n, c_n\}$.

2. If we can determine the time-evolution of these scattering data we will know the scattering data for all $t > 0$, $S(0) = \{b(k;t), \kappa_n(t), c_n(t)\}$

3. With this information we can solve the inverse scattering problem to find $u(x,t)$ for all $t > 0$.

We summarize the procedure in the following diagram:

$$\begin{tikzcd}
u(x,0) \arrow[r, "\text{scattering}"] \arrow[d, dashed, "\text{KdV}"'] & S(0) \arrow[d, "\text{time-evolution}"] \\
u(x,t) & S(t) \arrow[l, "\text{inverse scattering}"]
\end{tikzcd}$$

It remains of course to see how the time-evolution of the scattering data can be determined. This is the question we now turn to.

4.2 Time evolution of the scattering data

We start with the Schrödinger equation depending on the extra parameter t

$$\psi'' + (\lambda - u)\psi = 0\,, \tag{4.12}$$

for $\psi(x;t)$, $\lambda(t)$ and $u(x,t)$ a solution to the KdV equation. Taking the x- and t-derivative of this equation we get

$$\psi''' - u'\psi + +(\lambda - u)\psi' = 0\,, \tag{4.13}$$
$$\dot\psi'' + (\dot\lambda - \dot u)\psi + (\lambda - u)\dot\psi = 0\,, \tag{4.14}$$

where $u(x,t)$ satisfies the Kdv equation

$$\dot u - 6uu' + u''' = 0\,. \tag{4.15}$$

It is convenient to define

$$R(x,t) = \dot\psi + u'\psi - 2(u + 2\lambda)\psi'\,. \tag{4.16}$$

We then find

$$(\psi' R - \psi R')' = \psi'' R - \psi R'' = -\psi[(\lambda - u)R - R'']\,, \tag{4.17}$$

where we used the Schrödinger equation. Using the definition of R, the Schrödinger equation and its derivatives derived above we find

$$(\psi' R - \psi R')' = -\psi^2[\dot u - 6uu' + u'' - \dot\lambda] = \dot\lambda\psi^2\,, \tag{4.18}$$

where we used the KdV equation in the last step. This equation determines the time-evolution of the spectrum and scattering data as we will now show.

The discrete spectrum

Taking $\lambda = -\kappa_n^2$ and $\psi = \psi_n$ in (4.18) and integrating over all x we find

$$[\psi_n' R_n - \psi_n R_n']_{-\infty}^{\infty} = -\frac{d}{dt}(\kappa_n^2)\int_{-\infty}^{\infty}\psi_n^2 dx = -\frac{d}{dt}(\kappa_n^2)\,, \tag{4.19}$$

where R_n denotes R with the discrete eigenfunction ψ_n substituted and we used the fact that these are normalized in the last step. Because ψ_n and therefore also R_n decays exponentially as $|x| \to \infty$ we find

$$\frac{d}{dt}(\kappa_n^2) = 0 \qquad \Rightarrow \qquad \kappa_n = \text{constant}\,. \tag{4.20}$$

The discrete eigenvalues are *constants of motion*, once they are determined from the initial potential (profile) $u(x,0)$ they remain fixed for all t!

We also need to determine the time-evolution of the normalization constants $c_n(t)$. Since κ_n is constant (4.18) says that

$$(\psi_n' R_n - \psi_n R_n')' = 0 \tag{4.21}$$

which we can integrate to give

$$\psi_n' R_n - \psi_n R_n' = g_n(t)\,, \tag{4.22}$$

for some arbitrary functions $g_n(t)$. But since $\psi_n, R_n \to 0$ as $|x| \to \infty$ we find $g_n(t) = 0$. Thus

$$\psi_n' R_n - \psi_n R_n' = 0\,, \tag{4.23}$$

or dividing by ψ_n^2,

$$\left(\frac{R_n}{\psi_n}\right)' = 0 \qquad \Rightarrow \qquad \frac{R_n}{\psi_n} = h_n(t)\,. \tag{4.24}$$

Multiplying by ψ_n^2 and integrating over all x gives

$$\begin{aligned} h_n(t) &= \int_{-\infty}^{\infty} \psi_n R_n dx = \int_{-\infty}^{\infty} \psi_n(\dot{\psi}_n + u'\psi_n - 2(u - 2\kappa_n^2)\psi_n')dx \\ &= \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} \psi_n^2 dx + \int_{-\infty}^{\infty} (u\psi_n^2 - 2\psi_n'^2 + 4\kappa_n^2\psi_n^2)' dx = 0 \end{aligned} \tag{4.25}$$

where we used the normalization of ψ_n the Schrödinger equation and the fact that $\psi_n \to 0$ as $|x| \to \infty$. Therefore we have

$$0 = R_n = \dot{\psi}_n + u'\psi_n - 2(u - 2\kappa_n^2)\psi_n'\,, \tag{4.26}$$

which determines the time-evolution of the discrete eigenfunctions ψ_n. Using the fact that $u, u' \to 0$ and

$$\psi_n \sim c_n e^{-\kappa_n x} \tag{4.27}$$

as $x \to +\infty$ we finally find

$$\dot{c}_n - 4\kappa_n^3 c_n = 0 \qquad \Rightarrow c_n(t) = c_n(0)e^{4\kappa_n^3 t}\,, \tag{4.28}$$

where $c_n(0)$ $(n = 1, \ldots, N)$ are the normalization constants at $t = 0$.

The continuous spectrum

We can apply the same procedure to this case, for which $\lambda = k^2$. Since k can take any real value we are allowed to consider the time-evolution with k fixed. With this choice, taking $\psi = \hat{\psi}$ (4.18) gives

$$(\hat{\psi}'\hat{R} - \hat{\psi}\hat{R}')' = 0 \qquad \Rightarrow \qquad \hat{\psi}'\hat{R} - \hat{\psi}\hat{R}' = g(t;k)\,, \tag{4.29}$$

for some function $g(t;k)$. Since

$$\hat{\psi} \sim a e^{-ikx} \qquad x \to -\infty \tag{4.30}$$

we find from the definition of R that

$$\hat{R} \sim \dot{\hat{\psi}} - 4k^2\hat{\psi}' \sim (\dot{a} + 4ik^3 a)e^{-ikx} \qquad x \to -\infty\,, \tag{4.31}$$

and it follows that $g(t;k) = 0$. Therefore

$$\hat{\psi}'\hat{R} - \hat{\psi}\hat{R}' = 0 \qquad \Rightarrow \qquad \frac{\hat{R}}{\hat{\psi}} = h(t;k)\,. \tag{4.32}$$

Looking at the behavior as $x \to -\infty$ we find

$$\dot{a} + 4ik^3 a = h(t;k)a\,, \tag{4.33}$$

while as $x \to +\infty$ we have

$$\hat{\psi} \sim e^{-ikx} + be^{ikx} \tag{4.34}$$

and

$$\hat{R} \sim \dot{b}e^{ikx} + 4ik^3(e^{-ikx} - be^{ikx}) \tag{4.35}$$

and we get the condition

$$\hat{\psi} \sim e^{-ikx} + be^{ikx} \tag{4.36}$$

and

$$\dot{b}e^{ikx} + 4ik^3(e^{-ikx} - be^{ikx}) = h(e^{-ikx} + be^{ikx})\,. \tag{4.37}$$

The coefficients of both exponentials have to match and we find

$$h = 4ik^3 \qquad \Rightarrow \qquad \dot{a} = 0 \tag{4.38}$$

and

$$\dot{b} - 8ik^3 b = 0 \qquad \Rightarrow \qquad b(k;t) = b(k;0)e^{8ik^3 t}\,. \tag{4.39}$$

This completes the derivation of the time-evolution of the scattering data. The important ones for the inverse scattering problem are

$$\kappa_n = \text{constant}\,, \qquad c_n(t) = c_n(0)e^{4\kappa_n^3 t} \qquad (n = 1, \ldots, N) \tag{4.40}$$

$$b(k;t) = b(k;0)e^{8ik^3 t}\,. \tag{4.41}$$

4.3 Summary of the solution

Before analyzing examples of solutions we will summarize here the solution to the initial value problem for the KdV equation via what is usually called the *inverse scattering transform*. We want to solve the equation

$$\dot{u} - 6uu' + u''' = 0\,, \tag{4.42}$$

with the initial profile $u(x,0) = f(x)$ given. We assume that f is sufficiently well-behaved to ensure the existence of a solution of the KdV equation and also of the Schrödinger (Sturm-Liouville) equation

$$\psi'' + (\lambda - u)\psi = 0\,. \tag{4.43}$$

We first solve this equation for $u(x,0) = f(x)$ to determine the scattering data, in particular the discrete spectrum $-\kappa_n^2$ ($n = 1, \ldots, N$), normalization constants $c_n(0)$ and reflection coefficient $b(k;0)$. The time-evolution of these data are then given by

$$\kappa_n = \text{constant}\,, \qquad c_n(t) = c_n(0)e^{4\kappa_n^3 t}\,, \qquad b(k;t) = b(k;0)e^{8ik^3 t}\,. \tag{4.44}$$

The function entering the Marchenko equation is

$$F(x;t) = \sum_{n=1}^{N} c_n^2(0)e^{8\kappa_n^3 t - \kappa_n x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k;0)e^{8ik^3 t + ikx}dk\,, \tag{4.45}$$

which now depends also on the parameter t. The Marchenko equation is

$$K(x,z;t) + F(x+z;t) + \int_x^\infty K(x,y;t)F(y+z;t)dy = 0\,. \tag{4.46}$$

Finally the solution to the KdV equation is expressed in terms of the solution to this equation as

$$u(x,t) = -2\frac{\partial}{\partial x}K(x,x;t)\,. \tag{4.47}$$

Note that the solution outlined here involves two potentially difficult steps: solving the scattering problem for the initial profile $u(x,0) = f(x)$ and solving the Marchenko equation. Nevertheless we have managed to reduce solving a non-linear PDE to solving two linear problems, a second order ODE and an ordinary integral equation. We will now consider some simple examples of solutions to illustrate how the method works.

4.4 Reflectionless potentials

To illustrate how the inverse scattering transform works we will take the initial profile $u(x,0)$ to be a sech^2 function. In fact we will take the coefficient in front to correspond to a reflectionless potential, i.e. one that leads to $b(k) = 0$. We showed in the previous chapter when we analyzed the scattering problem that this is the case for

$$u(x,0) = -N(N+1)\operatorname{sech}^2 x\,. \tag{4.48}$$

We will see that this choice of initial profile describes the *N-soliton solution.*

$N = 1$: The solitary wave

In this case the initial profile is

$$u(x,0) = -2\operatorname{sech}^2 x\,. \tag{4.49}$$

The scattering problem is

$$\psi'' + (\lambda + 2\operatorname{sech}^2 x)\psi = 0\,. \tag{4.50}$$

When analyzing the scattering problem for a sech^2 potential we showed that introducing $T = \tanh x$ this becomes the associated Legendre equation

$$\frac{d}{dT}\left((1-T^2)\frac{d\psi}{dT}\right) + \left(2 + \frac{\lambda}{1-T^2}\right)\psi = 0\,. \tag{4.51}$$

This is the equation we solve to determine spherical harmonics (for $\ell = 1$). For $\lambda = -\kappa^2$ bounded solutions exist only for $\kappa = m$ with $m \leq \ell$ an integer. Setting $\ell = N = 1$ we find that there is only one discrete eigenvalue, $\kappa_1 = 1$. The corresponding eigenfunction is proportional to the associated Legendre function

$$\psi_1(x) \propto P_1^1(\tanh x) = -\operatorname{sech} x \tag{4.52}$$

and since

$$\int_{-\infty}^{\infty} \operatorname{sech}^2 x dx = 2 \tag{4.53}$$

the normalized eigenfunction is (the sign is irrelevant)

$$\psi_1(x) = \frac{1}{\sqrt{2}} \operatorname{sech} x \,. \tag{4.54}$$

Its asymptotic behavior is

$$\psi_1(x) \sim \sqrt{2} e^{-x} \qquad \text{as} \qquad x \to +\infty \,, \tag{4.55}$$

and we read off that $c_1(0) = \sqrt{2}$. Therefore

$$c_1(t) = \sqrt{2} e^{4t} \,. \tag{4.56}$$

This is all we need since we already know from our previous analysis that $b(k) = 0$. We find

$$F(x;t) = c_1^2(t) e^{-\kappa_1 x} = 2e^{8t-x} \,. \tag{4.57}$$

The Marchenko equation becomes

$$K(x,z;t) + 2e^{8t-(x+z)} + 2\int_x^\infty K(x,y;t) e^{8t-(y+z)} dy = 0 \,. \tag{4.58}$$

Looking at the z-dependence we see that the solution is of the form

$$K(x,z;t) = L(x,t) e^{-z} \tag{4.59}$$

and plugging this in we find

$$0 = L(x,t) + 2e^{8t-x} + 2L(x,t)e^{8t} \int_x^\infty e^{-2y} dy = L(x,t) + 2e^{8t-x} + e^{8t-2x} L(x,t) \,, \tag{4.60}$$

so that

$$L(x,t) = \frac{-2e^{8t-x}}{1+e^{8t-2x}} \,. \tag{4.61}$$

Finally we get

$$u(x,t) = -2\frac{\partial}{\partial x}\left(\frac{-2e^{8t-x}}{1+e^{8t-2x}} e^{-x}\right) = 4\frac{\partial}{\partial x}\left(\frac{1}{1+e^{2x-8t}}\right) = -2\operatorname{sech}^2(x-4t) \,. \tag{4.62}$$

This is the solitary wave solution of amplitude -2 and speed of propagation 4. We have already found this solution by different methods, looking for a traveling wave of fixed form. However, the following solutions would be very difficult to find by other methods.

$N = 2$: two-soliton solution

In this case the initial profile is

$$u(x,0) = -6\operatorname{sech}^2 x \tag{4.63}$$

and we need to solve the scattering problem

$$\psi'' + (\lambda + 6\operatorname{sech}^2 x)\psi = 0 \,. \tag{4.64}$$

As before we transform this to the associated Legendre equation by introducing the variable $T = \tanh x$. Now there are two discrete eigenvalues with $\kappa_1 = 1$ and $\kappa_2 = 2$. The normalized eigenfunctions are

$$\psi_1(x) = \sqrt{\frac{3}{2}} \tanh x \operatorname{sech} x\,, \qquad \psi_2(x) = \frac{\sqrt{3}}{2} \operatorname{sech}^2 x\,. \tag{4.65}$$

Their asymptotic behavior is

$$\psi_1(x) \sim \sqrt{6} e^{-x},\,, \qquad \psi_2(x) \sim 2\sqrt{3} e^{-2x} \qquad \text{as} \qquad x \to +\infty\,, \tag{4.66}$$

and we read off $c_1(0) = \sqrt{6}$ and $c_2(0) = 2\sqrt{3}$. Their time-dependence is then given by

$$c_1(t) = \sqrt{6} e^{4t}\,, \qquad c_2(t) = 2\sqrt{3} e^{32t}\,. \tag{4.67}$$

As before our choice of profile ensures that $b(k;t) = 0$. The function F becomes

$$F(x;t) = 6e^{8t-x} + 12e^{64t-2x} \tag{4.68}$$

and the Marchenko equation is

$$K(x,z;t) + 6e^{8t-(x+z)} + 12e^{64t-2(x+z)} + \int_x^\infty K(x,y;t) \left[6e^{8t-(y+z)} + 12e^{64t-2(y+z)}\right] dy = 0\,. \tag{4.69}$$

It is clear that the solution has the form

$$K(x,z;t) = L_1(x,t)e^{-z} + L_2(x,t)e^{-2z}\,, \tag{4.70}$$

since F is a separable function. Collecting the coefficients of e^{-z} and e^{-2z} gives the pair of equations

$$L_1 + 6e^{8t-x} + 6e^{8t} \left(L_1 \int_x^\infty e^{-2y} dy + L_2 \int_x^\infty e^{-3y} dy \right) = 0\,, \tag{4.71}$$

$$L_2 + 12e^{64t-2x} + 12e^{64t} \left(L_1 \int_x^\infty e^{-3y} dy + L_2 \int_x^\infty e^{-4y} dy \right) = 0\,. \tag{4.72}$$

Evaluating the integrals they become

$$L_1 + 6e^{8t-x} + 3L_1 e^{8t-2x} + 2L_2 e^{8t-3x} = 0\,, \tag{4.73}$$

$$L_2 + 12e^{64t-2x} + 4L_1 e^{64t-3x} + 3L_2 e^{64t-4x} = 0\,, \tag{4.74}$$

with solution

$$L_1 = 6\frac{e^{72t-5x} - e^{8t-x}}{D}\,, \qquad L_1 = -12\frac{e^{72t-4x} + e^{64t-2x}}{D}\,, \tag{4.75}$$

where

$$D = 1 + 3e^{8t-2x} + 3e^{64t-4x} + e^{72t-6x}\,. \tag{4.76}$$

Finally the solution to the KdV equation becomes

$$u(x,t) = -2\frac{\partial}{\partial x}(L_1 e^{-x} + L_2 e^{-2x}) = 12\frac{\partial}{\partial x}\left(\frac{e^{8t-2x} + e^{72t-6x} - 2e^{64t-4x}}{D}\right) \tag{4.77}$$

and after a bit of algebra one finds

$$u(x,t) = -12\frac{3+4\cosh(2x-8t)+\cosh(4x-64t)}{(3\cosh(x-28t)+\cosh(3x-36t))^2}\,. \tag{4.78}$$

We will now analyze this solution a bit to see why it is called the two-soliton solution. The first thing to note is that the solution is valid for all t so we can consider how it evolves into the initial profile at $t=0$ and one from there.

The solution is shown at different times in figure 4.1. For large negative t is describes two (almost) solitary waves. The taller one is to the left and since it moves faster it catches up with the shorter one to form a single wave – our initial profile at $t=0$. Finally the taller wave overtakes the shorter one. Superficially this looks like a linear process but careful examination shows that the taller wave has moved forward and the shorter wave backward relative to where they would be if they just passed each other.

Each solitary wave appearing at $t\to\pm\infty$ and interacting in this way, via a *phase shift*, is called a *soliton*. The solution we have described is therefore called the *two-soliton* solution.

We can gain a better understanding of the solution by looking at its asymptotics as $t\to\pm\infty$. Writing $\xi = x-16t$ the solution takes the form

$$u(x,t) = -12\frac{3+4\cosh(2\xi+24t)+\cosh(4\xi)}{(3\cosh(\xi-12t)+\cosh(3\xi+12t))^2}\,. \tag{4.79}$$

We can now take $t\to\pm\infty$ with ξ held fixed, which means that we are following the wave with speed 16. We get

$$\begin{aligned} u(x,t) \sim& -96\frac{e^{\pm(2\xi+24t)}}{(3e^{\mp(\xi-12t)}+e^{\pm(3\xi+12t)})^2} = -96\frac{1}{(3e^{\mp 2\xi}+e^{\pm 2\xi})^2} \\ =& -32\frac{1}{(e^{\mp 2\xi+\ln 3/2}+e^{\pm 2\xi-\ln 3/2})^2} = -8\operatorname{sech}^2\left(2\xi\mp\tfrac{1}{2}\ln 3\right) \qquad \text{as} \qquad t\to\pm\infty\,. \end{aligned} \tag{4.80}$$

Doing the same for the wave of speed 4 we can add the two together to get the solution (the error is exponentially small since the waves are far apart at very early/late times)

$$u(x,t) \sim -8\operatorname{sech}^2\left(2\xi\mp\tfrac{1}{2}\ln 3\right) - 2\operatorname{sech}^2\left(\eta\pm\tfrac{1}{2}\ln 3\right) \qquad \text{as} \qquad t\to\pm\infty\,, \tag{4.81}$$

where $\xi = x-16t$ and $\eta = x-4t$. Here we see that the taller wave of amplitude -8 is shifted to the right by $\frac{1}{2}\ln 3$ while the shorter one of amplitude -2 is shifted to the left by $\ln 3$ by the interaction.

N-soliton solution

We now want to analyze the case of general N. The initial profile is

$$u(x,0) = -N(N+1)\operatorname{sech}^2 x \tag{4.82}$$

There are now N discrete eigenvalues and no contribution from the continuous spectrum since $b(k)=0$. The discrete eigenvalues are $\lambda = -\kappa_n^2$ with $\kappa_n = n$ and $n=1,2,\ldots,N$. The normalized eigenfunctions go as

$$\psi_n(x) \sim c_n e^{-nx} \qquad \text{as} \qquad x\to+\infty\,. \tag{4.83}$$

They are proportional to the associated Legendre functions

$$\psi_n(x) \propto P_N^n(\tanh x) \tag{4.84}$$

and $c_n(0)$ is determined from the normalization condition. Then

$$c_n(t) = c_n(0)e^{4n^3 t}\,. \tag{4.85}$$

The function F becomes

$$F(x;t) = \sum_{n=1}^{N} c_n^2(0)e^{8n^3t-nx} \tag{4.86}$$

and the Marchenko equation becomes

$$K(x,z;t) + \sum_{n=1}^{N} c_n^2(0)e^{8n^3t-n(x+z)} + \int_x^\infty K(x,y;t)\sum_{n=1}^{N} c_n^2(0)e^{8n^3t-n(y+z)}dy = 0\,. \tag{4.87}$$

The solution must take the form

$$K(x,z;t) = \sum_{n=1}^{N} L_n(x,t)e^{-nz}\,. \tag{4.88}$$

In section 3.3 we analyzed the solution of the Marchenko equation for a general separable F (i.e. $F(x+z) = \sum_{n=1}^N X_n(x)Z_n(z)$). We can apply what we learned there to the present case. The Marchenko equation reduces to the algebraic system

$$AL + X = 0 \tag{4.89}$$

with

$$L = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_N \end{pmatrix}, \qquad X = \begin{pmatrix} c_1^2(0)e^{8t-x} \\ c_2^2(0)e^{64t-2x} \\ \vdots \\ c_N^2(0)e^{8N^3t-Nx} \end{pmatrix}, \tag{4.90}$$

The $N\times N$ matrix A has elements

$$A_{mn} = \delta_{mn} + \frac{c_m^2(0)}{m+n}e^{8m^3t-(m+n)x} \tag{4.91}$$

and we showed that the solution to the KdV equation takes the form

$$u(x,t) = -2\frac{\partial^2}{\partial x^2}\log\det A\,. \tag{4.92}$$

As in the 2-soliton example we can determine the asymptotic form of the solution. Setting $\xi_n = x - 4\kappa_n^2 t = x - 4n^2 t$ and taking $t \to \pm\infty$ with ξ_n held fixed the behavior of u is

$$u(x,t) \sim -2n^2\,\mathrm{sech}^2(n(x-4n^2t) \mp x_n) \qquad \text{as} \qquad t \to \pm\infty\,, \tag{4.93}$$

where the phase, x_n, is given by

$$e^{2x_n} = \prod_{m=1,\,m\neq n}^{N} \left|\frac{n-m}{n+m}\right|^{\mathrm{sgn}(n-m)} \qquad n = 1,2,\ldots,N\,. \tag{4.94}$$

The N-soliton form of the solution is evident from the full asymptotic solution

$$u(x,t) \sim -2\sum_{n=1}^{N} n^2 \operatorname{sech}^2(n(x-4n^2t) \mp x_n) \qquad \text{as} \qquad t \to \pm\infty\,. \tag{4.95}$$

The asymptotic solution consists of N separate solitons ordered according to their speeds: as $t \to +\infty$ the tallest (fastest) is at the front followed by progressively shorter waves. All N solitons interact to form a single sech^2 pulse at $t = 0$ – the initial profile $u = -N(N+1)\operatorname{sech}^2 x$.

4.5 Solutions with $b(k) \neq 0$

So far we have considered reflectionless initial profiles. It is clear that a more general choice of $u(x,0)$ will however lead to to $b(k) \neq 0$. Unfortunately when $b(k) \neq 0$ it is not possible to solve the Marchenko equation in closed form. Instead we have to resort to numerical and asymptotic analysis of the solution. We will now describe the features of the solutions for some simple initial profiles.

Example 1: delta-function initial profile

We take the initial profile to be

$$u(x,0) = -U_0\delta(x)\,, \tag{4.96}$$

with $U_0 > 0$ and constant. We know from our analysis of the scattering problem that there is a single discrete eigenvalue $\lambda = -\kappa_1^2$ with $\kappa_1 = \frac{1}{2}U_0$ and eigenfunction

$$\psi_1(x) = \sqrt{\kappa_1}e^{\mp\kappa_1 x} \qquad x \gtrless 0\,. \tag{4.97}$$

The reflection coefficient is given by

$$b(k) = -\frac{U_0}{U_0 + 2ik}\,. \tag{4.98}$$

The time-evolution of these scattering data is

$$c_1(t) = \sqrt{\kappa_1}e^{4\kappa_1^3 t}\,, \qquad b(k;t) = -\frac{U_0 e^{8ik^3t}}{U_0+2ik}\,. \tag{4.99}$$

The function F becomes

$$F(x;t) = \kappa_1 e^{8\kappa_1^3 t - \kappa_1 x} - \frac{U_0}{2\pi}\int_{-\infty}^{\infty} \frac{e^{8ik^3t+ikx}}{U_0+2ik}dk\,, \tag{4.100}$$

but now the Marchenko equation cannot be solved completely. However, it is clear that the solution should incorporate the single soliton associated with the discrete eigenvalue $\kappa_1 = \frac{1}{2}U_0$, i.e.

$$u(x,t) \sim -\tfrac{1}{2}U_0^2\operatorname{sech}^2\left[\tfrac{1}{2}U_0(x-U_0^2t-x_1)\right]\,, \qquad x_1 = -\frac{\ln 2}{U_0}\,. \tag{4.101}$$

This solution will be valid where the integral term in F is zero (or very small). To see the role of the integral term in F we take $t \to +\infty$ and consider the region $x < 0$ where the soliton solution is exponenially small. Writing $k = l\sqrt{x/t}$ the integral becomes

$$-\frac{U_0}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\sqrt{x^3/t}(8l^3+l)}}{U_0\sqrt{t/x}+2il}dl = -\frac{U_0}{2\pi}\int_{-\infty}^{\infty}\frac{e^{iy^{3/2}(8l^3+l)}}{U_0t^{1/3}y^{-1/2}+2il}dl \tag{4.102}$$

where we introduced $y = xt^{-1/3}$. Taking now $t \to +\infty$ with y held fixed gives an oscillatory dispersive wave propagating to the left with amplitude decaying as $t^{-1/3}$ as $t \to +\infty$. The solution is depicted in figure 4.2.

If instead we would take $U_0 < 0$ there is no discrete eigenvalue and the solution has no soliton component, only a dispersive wave train for $t > 0$.

Example 2: Negative sech^2 initial profile

Consider the initial profile

$$u(x,0) = -4\,\mathrm{sech}^2\, x\,. \tag{4.103}$$

The number of discrete eigenvalues is

$$\left[\sqrt{U_0 + \frac{1}{4}} - \frac{1}{2}\right] + 1 = 2\,, \tag{4.104}$$

for $U_0 = 4$, where $[\cdots]$ denotes the integer part, so we will get a solution with two solitons. But since 4 cannot be written as $N(N+1)$ the solution will include a dispersive component. The solution is depicted in figure 4.3.

Example 3: Positive sech^2 initial profile

Whenever $u(x,0) > 0$ there are no discrete eigenvalues (there are no bound states for a positive potential) and the solution will develop without the emergence of a soliton. The initial profile collapses into a wave train which disperses to the left. The solution for the profile

$$u(x,0) = \mathrm{sech}^2\, x \tag{4.105}$$

is depicted in figure 4.4. The solution can be seen to approach the (inverted) Airy function.

Here we will leave the discussion of explicit solutions of the KdV equation. Instead we want to now address the question whether there are other equations that can also be solve in a similar way. This will lead us to develop some new and powerful tools.

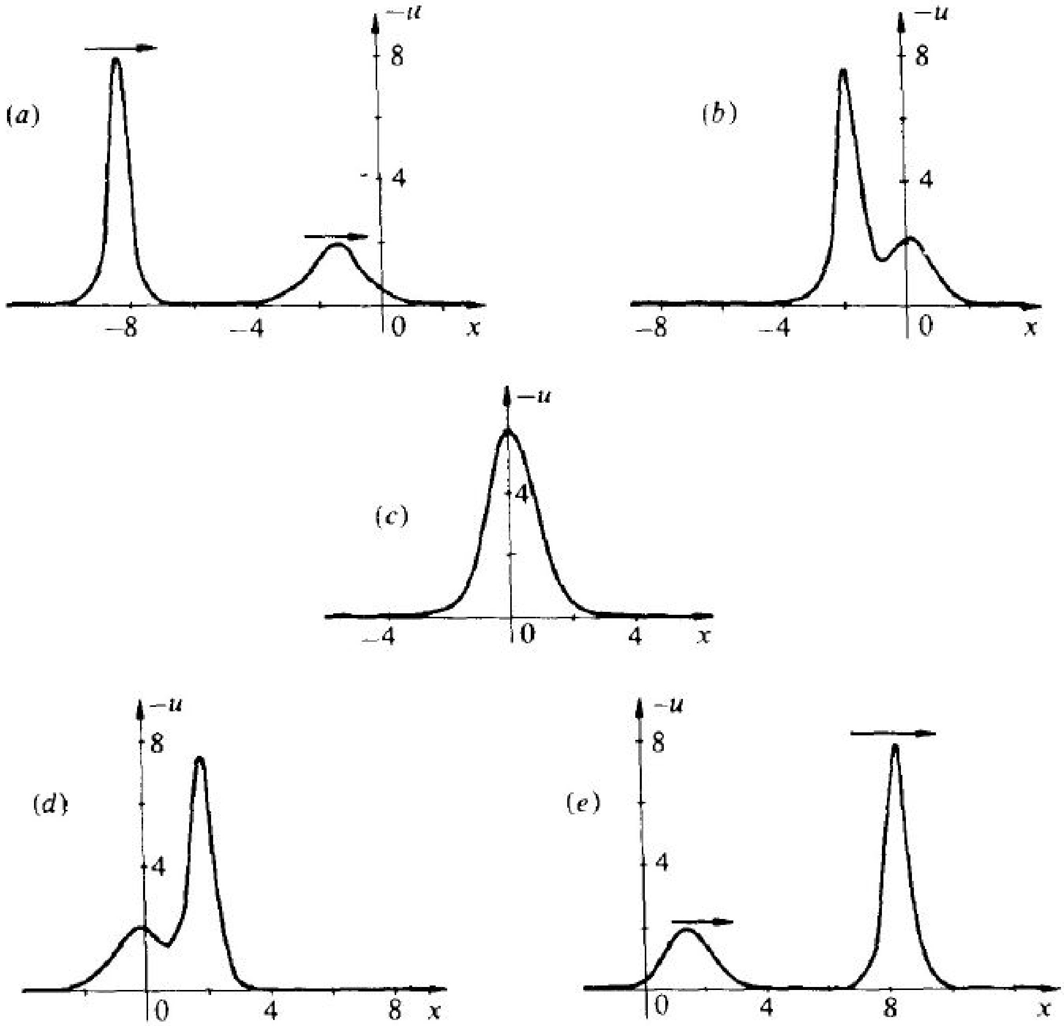

Figure 4.1: The two-soliton solution at various times: (a) $t = -0.5$ (b) $t = -0.1$ (c) $t = 0$ (d) $t = 0.1$ (e) $t = 0.5$ [Figure taken from Drazin & Johnson.]

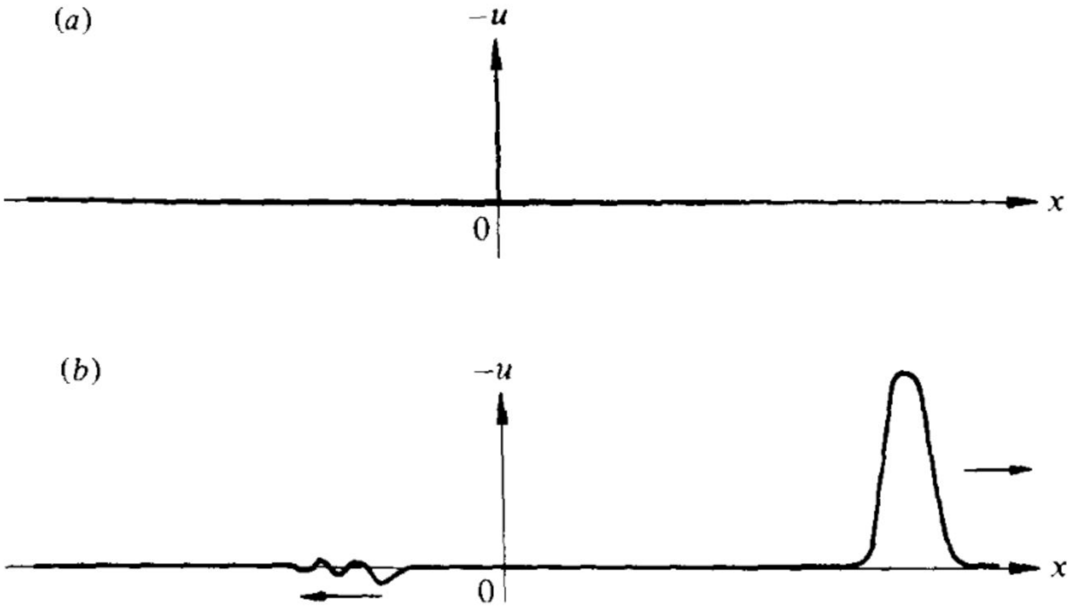

Figure 4.2: The delta-function initial profile: (a) inital profile (b) solution at a later time. [Figure taken from Drazin & Johnson.]

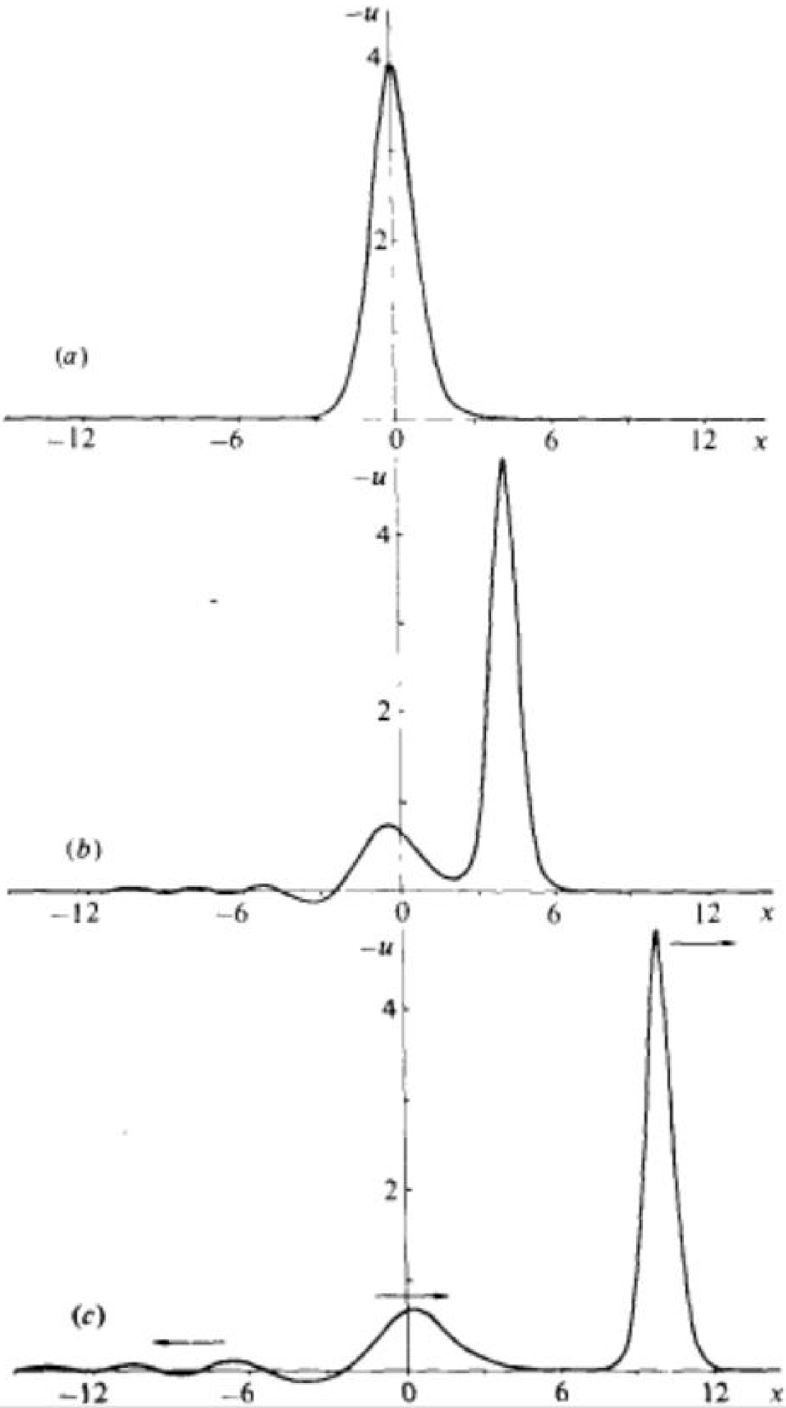

Figure 4.3: Solution with two solitons and dispersive wave, inital profile $u(x, 0) = -4\,\mathrm{sech}^2\, x$: (a) $t = 0$ (b) $t = 0.4$ (c) $t = 1$ [Figure taken from Drazin & Johnson.]

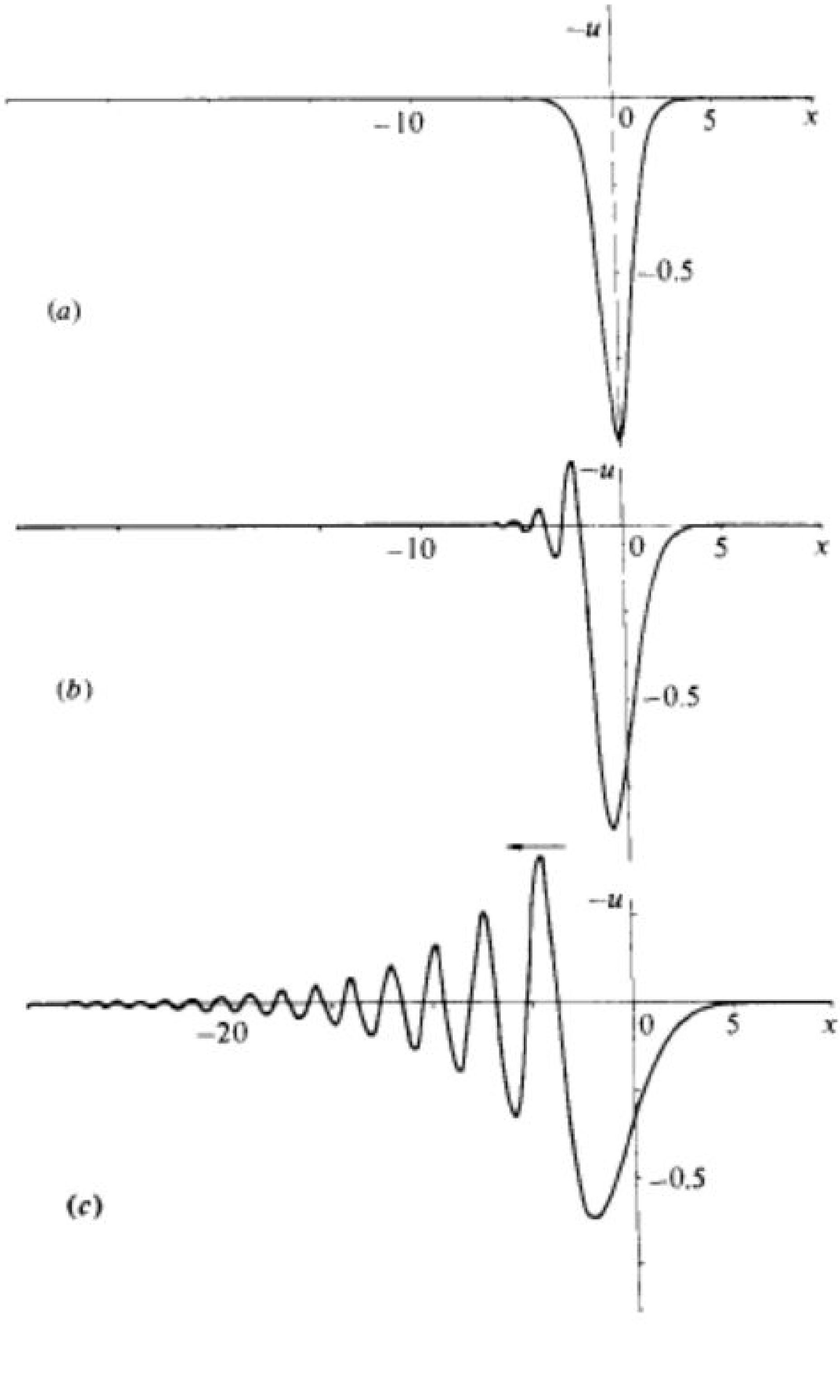

Figure 4.4: Solution with dispersive wave only, initial profile $u(x,0) = \operatorname{sech}^2 x$: (a) $t = 0$ (b) $t = 0.1$ (c) $t = 0.5$ [Figure taken from Drazin & Johnson.]

Chapter 5

Lax formulation and more general inverse methods

So far we have described the solution of the KdV equation via the inverse scattering transform. To understand if there are other problems that can be solved in a similar way we need to develop some more general tools. A very important tool was introduced by Lax in 1968.

5.1 Lax formulation

Suppose we are interested in an evolution equation of the form

$$\dot{u} = N(u)\,, \tag{5.1}$$

where N is some non-linear operator this is independent of t but may involve x and partial derivatives with respect to x. For example, in the KdV case

$$N(u) = 6uu' - u'''\,. \tag{5.2}$$

Now suppose that the evolution equation (5.1) can be expressed in the form

$$\dot{L} = [M, L] = ML - LM\,, \tag{5.3}$$

where L and M are some operators linear in x, which can depend on $u(x,t)$ and which operate on some auxiliary Hilbert space H. Note that

$$\dot{L} = \frac{\partial}{\partial t} L \tag{5.4}$$

means the derivative with respect to t that appear explicitly in L, e.g. for

$$L = -\frac{\partial^2}{\partial x^2} + u(x,t) \qquad \text{we have} \qquad \dot{L} = \dot{u}\,. \tag{5.5}$$

The Hilbert space has an inner product (ϕ, ψ) and we assume that L is self-adjoint (Hermitian), i.e. $(L\phi, \psi) = (\phi, L^{\dagger}\psi) = (\phi, L\psi)\ \forall \phi, \psi \in H$ (cf. the Hamiltonian in quantum mechanics).

Consider now the *spectral problem* for L, i.e. the eigenvalue equation

$$L\psi = \lambda\psi\,, \tag{5.6}$$

where $\lambda = \lambda(t)$. Taking a t-derivative we find

$$\dot{L}\psi + L\dot{\psi} = \dot{\lambda}\psi + \lambda\dot{\psi} \tag{5.7}$$

and using $\dot{L} = [M, L]$ this becomes

$$\dot{\lambda}\psi = (L-\lambda)\dot{\psi} + [M, L]\psi = (L-\lambda)(\dot{\psi} - M\psi)\,, \tag{5.8}$$

where we used again the fact that $L\psi = \lambda\psi$. Taking the inner product with ψ we find

$$\dot{\lambda}(\psi,\psi) = (\psi, (L-\lambda)[\dot{\psi} - M\psi]) = ((L-\lambda)\psi, \dot{\psi} - M\psi) = 0\,, \tag{5.9}$$

since L is self-adjoint. Therefore we conclude that the eigenvalues of the operator L are constant. Equation (5.8) now gives

$$(L-\lambda)(\dot{\psi} - M\psi) = 0\,, \tag{5.10}$$

so that $\dot{\psi} - M\psi$ is also an eigenfunction of L with eigenvalue λ. Therefore we must have (assuming a non-degenerate case)

$$\dot{\psi} - M\psi \propto \psi\,. \tag{5.11}$$

But we can always shift M by a function of t times the identity operator since $[M, L]$ remains invariant, so we may take

$$\dot{\psi} = M\psi\,, \tag{5.12}$$

which gives the time-evolution of the eigenfunction ψ.

We have shown that if the evolution equation

$$\dot{u} = N(u) \tag{5.13}$$

can be expressed as the *Lax equation*

$$\dot{L} + [L, M] = 0\,, \tag{5.14}$$

where L, M are referred to as the *Lax pair*, the spectrum of L

$$L\psi = \lambda\psi \tag{5.15}$$

evolves according to

$$\lambda = \text{constant}\,, \qquad \dot{\psi} = M\psi\,. \tag{5.16}$$

There is a converse to this statement. The two equations

$$L\psi = \lambda\psi\,, \qquad \dot{\psi} = M\psi\,, \tag{5.17}$$

with λ a constant, imply, taking the time-derivative of the first and using the equations, that

$$0 = \dot{L}\psi + L\dot{\psi} - \lambda\dot{\psi} = \dot{L}\psi + LM\psi - \lambda M\psi = (\dot{L} + [L, M])\psi\,. \tag{5.18}$$

This means that the Lax equation is the *compatibility condition* for the system of equations (5.17). This is often a useful way of thinking about the Lax equation.

5.2 Lax formulation of the KdV equation

For the KdV equation it is natural to assume that we should take

$$L = -\frac{\partial^2}{\partial x^2} + u\,, \tag{5.19}$$

so that $L\psi = \lambda\psi$ becomes the Schrödinger (Sturm-Liouville) equation. We still need to find M, if it exists. Since L is hermitian (self-adjoint), $L^\dagger = L$, we have

$$[M, L] = \dot{L} = \dot{L}^\dagger = [M, L]^\dagger = LM^\dagger - M^\dagger L = -[M^\dagger, L]\,, \tag{5.20}$$

and we see that we should take M to be anti-hermitian, $M^\dagger = -M$. Therefore we should construct M from a combination of odd derivatives in x. (For example

$$(\frac{\partial^n}{\partial x^n}\phi, \psi) = \int_{-\infty}^{\infty} \frac{\partial^n \phi}{\partial x^n}\psi = (-1)^n \int_{-\infty}^{\infty} \phi \frac{\partial^n \psi}{\partial x^n} = (-1)^n (\phi, \frac{\partial^n}{\partial x^n}\psi)\,, \tag{5.21}$$

where we partially integrated n times using the fact that $\phi, \phi', \ldots, \psi, \psi', \ldots \to 0$ as $x \to \pm\infty$, i.e. $\frac{\partial^n}{\partial x^n}$ is anti-hermitian for n odd.)

The simplest possibility is to take a single derivative

$$M = c\frac{\partial}{\partial x}\,. \tag{5.22}$$

Taking c to be a constant we find

$$[L, M] = \left[-\frac{\partial^2}{\partial x^2} + u, c\frac{\partial}{\partial x}\right] = -cu'\,. \tag{5.23}$$

Therefore the Lax equation becomes

$$0 = \dot{L} + [L, M] = \dot{u} - cu'\,, \tag{5.24}$$

which is the simplest linear wave equation. This is not very useful since we can solve this equation directly. The next possibility is to take M to be a third order operator

$$M = -\alpha\frac{\partial^3}{\partial x^3} + U\frac{\partial}{\partial x} + \frac{\partial}{\partial x}U + A\,, \tag{5.25}$$

where α is a constant, $U = U(x, t)$ and we wrote the $\frac{\partial}{\partial x}$-terms in a form that will be convenient later. In this case we find

$$[L, M] = \alpha u''' - U''' - A'' - 2u'U + \big(3\alpha u'' - 4U'' - 2A'\big)\frac{\partial}{\partial x} + \big(3\alpha u' - 4U'\big)\frac{\partial^2}{\partial x^2}\,. \tag{5.26}$$

Since this should equal $\dot{L} = \dot{u}$ which is a multiplicative operator the last two terms must vanish which happens if

$$U = \frac{3}{4}\alpha u \qquad \text{and} \qquad A = A(t)\,. \tag{5.27}$$

In this case the Lax equation becomes

$$0 = \dot{L} + [L, M] = \dot{u} - \frac{3}{2}\alpha u u' + \frac{1}{4}\alpha u'''\,, \tag{5.28}$$

which for $\alpha = 4$ is the KdV equation. In this case the operator M becomes

$$M = -4\frac{\partial^3}{\partial x^3} + 3u\frac{\partial}{\partial x} + 3\frac{\partial}{\partial x}u + A(t)\,, \tag{5.29}$$

so the evolution of ψ is

$$\dot{\psi} = M\psi = -4\psi''' + 6u\psi' + 3u'\psi + A\psi\,. \tag{5.30}$$

Using the Schrödinger equation in the first term this becomes

$$\dot{\psi} = 4[(\lambda - u)\psi]' + 6u\psi' + 3u'\psi + A\psi = 2(2\lambda + u)\psi' - u'\psi + A\psi\,. \tag{5.31}$$

With $A = 0$ we recover the time-evolution of the discrete eigenfunctions (4.26) and with $A = 4ik^3$ we recover that of the continuous ones.

5.2.1 The KdV hierarchy

The KdV equation appeared as our second example in the Lax formulation with L the Schrödinger operator. The procedure can be extended to higher-order non-linear equations. A little work shows that the appropriate choice of M is

$$M = -\alpha\frac{\partial^{2n+1}}{\partial x^{2n+1}} + \sum_{m=1}^{n}\left(U_m\frac{\partial^{2m-1}}{\partial x^{2m-1}} + \frac{\partial^{2m-1}}{\partial x^{2m-1}}U_m\right) + A\,, \tag{5.32}$$

where α is a constant, $U_m = U_m(x,t)$ and $A = A(t)$ (A does not enter the evolution equation). The condition that $[L, M]$ be a multiplicative operator imposes n conditions on the n unknown functions U_m, namely that the coefficient of $\frac{\partial^{2m-1}}{\partial x^{2m-1}}$ should vanish for $m = 1, 2, \ldots, n$. For $n = 1$ we recover the KdV equation while for $n = 2$ we get the equation

$$\dot{u} + 30u^2u' - 20u'u'' - 10uu''' + u''''' = 0\,, \tag{5.33}$$

a fifth order KdV equation.

We have now presented the first three evolution equations in the *KdV hierarchy*, each of which can be solved with the inverse scattering transform. This already gives us infinitely many non-linear integrable equations. The Lax formulation has provided this for free, so we see that it is indeed quite powerful. In fact it can me extended in various ways which makes it even more powerful. We will now consider another remarkable fact which follows from a slightly different version of the Lax formulation.

5.2.2 Infinitely many conservation laws

A remarkable property of integrable systems like the KdV equation is that they possess infinitely many conserved quantities. A nice way to see this uses a slightly different version of the Lax formulation.

Suppose that an evolution equation can be formulated as the *zero-curvature condition*

$$\dot{L} - M' + [L, M] = 0\,, \tag{5.34}$$

where now L and M are not operators but just matrices. However, they are required to depend non-trivially on an extra auxiliary *spectral parameter* λ, i.e. $L = L(x,t;\lambda)$ and $M = M(x,t;\lambda)$. Consider for example the choice

$$L = \begin{pmatrix} 0 & -1 \\ \lambda - u & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} -u' & -2(2\lambda + u) \\ 2(\lambda - u)(2\lambda + u) + u'' & u' \end{pmatrix}. \tag{5.35}$$

We then get the equation

$$0 = \dot{L} - M' + [L, M] = -\begin{pmatrix} 0 & 0 \\ \dot{u} - 6uu' + u''' & 0 \end{pmatrix}, \tag{5.36}$$

which is equivalent to the KdV equation.

The zero-curvature condition for L, M can again be thought of as a compatibility condition, this time for the two *linear* equations

$$V' = LV, \qquad \dot{V} = MV. \tag{5.37}$$

Indeed, taking the t-derivative of the first equation and subtracting the x-derivative of the second we find

$$0 = \frac{\partial}{\partial t}(LV) - \frac{\partial}{\partial x}(MV) = \dot{L}V + L\dot{V} - M'V - MV' = (\dot{L} - M' + [L, M])V. \tag{5.38}$$

For the case of the KdV equation these equations read

$$\begin{pmatrix} V_1' \\ V_2' \end{pmatrix} = \begin{pmatrix} -V_2 \\ (\lambda - u)V_1 \end{pmatrix}, \qquad \begin{pmatrix} \dot{V}_1 \\ \dot{V}_2 \end{pmatrix} = \begin{pmatrix} -u'V_1 - 2(2\lambda + u)V_2 \\ [2(\lambda - u)(2\lambda + u) + u'']V_1 + u'V_2 \end{pmatrix}. \tag{5.39}$$

Consider the equation for $\dot{V}_2$. It is

$$\dot{V}_2 = 2(\lambda - u)(2\lambda + u)V_1 + u''V_1 + u'V_2 = [2(2\lambda + u)V_2 + u'V_1]', \tag{5.40}$$

as is easily verified using the other equations. Writing $f = V_2$ and $g = 2(2\lambda + u)V_2 + u'V_1$ this equation reads

$$\dot{f} = g'. \tag{5.41}$$

This is a local conservation equation. It says that the integral of f over all x is a constant of the motion. But since L, M depend also on the auxiliary spectral parameter λ the same must be true for V_1, V_2 and therefore in particular $f = f(t, x; \lambda)$. This means that Taylor expanding f in powers of the spectral parameter each term is a conserved quantity. Therefore we have found infinitely many conservation laws for the KdV equation.

The Lax formulation is indeed powerful since it has given us both the KdV hierarchy and an infinite set of conserved quantities. We will now see how it can be exploited to construct more general methods for solving integrable equations.

5.3 The Zakharov-Shabat scheme

So far we have discussed the KdV equation and its solution via the inverse scattering transform in some detail. We are now finally ready to see how these methods can be generalized to other equations. In the early 70's two groups generalized the inverse scattering transform so that it could be applied to several problems. Zakharov and Shabat (ZS) generalized the Lax method while Ablowitz, Kaup, Newell and Segur (AKNS) generalized the Sturm-Liouville scattering problem. We will describe only the ZS scheme, which tends to be more useful, here.

5.3.1 Integral operators

The approach of ZS starts by defining three integral operators. Let $F(x,z)$ and $K_\pm(x,z)$ be $N\times N$-matrices with

$$\begin{array}{ll} K_+(x,z)=0 & \text{if} \quad z<x \\ K_-(x,z)=0 & \text{if} \quad z>x \end{array} \tag{5.42}$$

We define the integral operators J_F and $J_\pm$ by

$$J_F\psi=\int_{-\infty}^{\infty}F(x,z)\psi(z)dz\,, \qquad J_\pm\psi=\int_{-\infty}^{\infty}K_\pm(x,z)\psi(z)dz\,, \tag{5.43}$$

where ψ is an N-component vector. We further demand that these operators satisfy the identity

$$(I+J_+)(I+J_F)=I+J_- \tag{5.44}$$

where I is the $N\times N$ unit matrix. We also require that $I+J_+$ is invertible so that

$$I+J_F=(I+J_+)^{-1}(I+J_-)\,. \tag{5.45}$$

The identity (5.44) gives

$$(J_++J_F)\psi+J_+J_F\psi=J_-\psi \tag{5.46}$$

or

$$\int_{-\infty}^{\infty}(K_++F)(x,z)\psi(z)dz+\int_{-\infty}^{\infty}K_+(x,z)\left(\int_{-\infty}^{\infty}F(z,y)\psi(y)dy\right)dz=\int_{-\infty}^{\infty}K_-(x,z)\psi(z)dz\,. \tag{5.47}$$

Let's suppose that $\psi(z)=0$ for $z<x$. Then the RHS is zero and the double integral can be written (relabeling the integration variables)

$$\int_{-\infty}^{\infty}\int_{x}^{\infty}K_+(x,y)F(y,z)\psi(z)dydz\,. \tag{5.48}$$

Therefore we find

$$\int_{-\infty}^{\infty}\left(K_+(x,z)+F(x,z)+\int_{x}^{\infty}K_+(x,y)F(y,z)dy\right)\psi(z)dz=0\,. \tag{5.49}$$

Since this has to hold for all ψ such that $\psi(z)=0$ for $z<x$ we must have

$$K_+(x,z)+F(x,z)+\int_{x}^{\infty}K_+(x,y)F(y,z)dy=0 \qquad \text{for} \qquad z>x\,. \tag{5.50}$$

This is the matrix Marchenko equation. Similarly one can show by taking $z<x$ that

$$K_-(x,z)=F(x,z)+\int_{x}^{\infty}K_+(x,y)F(y,z)dy\,, \tag{5.51}$$

which determines K_- in terms of K_+ and F.

5.3.2 Differential operators

We now let $K_\pm$ and F depend on *two* auxiliary variables, e.g. t, y. The evolution of $K_\pm$ and F in t, y will be determined by appropriate linear differential operators. We define first the $N \times N$ matrix differential operator Δ_0 acting on $\psi(x; t, y)$, which has *constant coefficients* and commutes with J_F

$$[\Delta_0, J_F] = 0\,. \tag{5.52}$$

We also introduce an associated operator Δ defined by

$$\Delta(I + J_+) = (I + J_+)\Delta_0\,. \tag{5.53}$$

(The same holds with J_+ replaced by J_-.) Recall that $I + J_+$ is invertible so this equation indeed defines Δ. Δ_0 is sometimes referred to as 'undressed' and Δ as 'dressed' operator.

To see how this works consider the simple example, which will be useful later,

$$\Delta_0 = I\left(\alpha\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\,, \tag{5.54}$$

where I is the $N \times N$ unit matrix and α is a constant. We now get

$$\begin{aligned} 0 =&[\Delta_0, J_F]\psi = \Delta_0 J_F\psi - J_F\Delta_0\psi \\ =&\left(\alpha\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\int_{-\infty}^{\infty} F(x,z;t)\psi(z;t)dz - \int_{-\infty}^{\infty} F(x,z;t)\left(\alpha\frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2}\right)\psi(z;t)dz\,. \end{aligned} \tag{5.55}$$

Assuming $\psi, \frac{\partial}{\partial z}\psi \to 0$ as $z \to \pm\infty$, we may integrate by parts to get

$$\int_{-\infty}^{\infty}\left(\alpha\dot{F} - F'' + \frac{\partial^2}{\partial z^2}F\right)\psi dz = 0\,. \tag{5.56}$$

Since this holds for all ψ we find the linear wave equation for F

$$\alpha\dot{F} - F'' + \frac{\partial^2}{\partial z^2}F = 0\,. \tag{5.57}$$

The associated operator Δ is obtained from the definition in (5.53). Acting on ψ we get

$$\Delta\left(\psi(x;t) + \int_x^{\infty} K_+(x,z;t)\psi(z;t)dz\right) = \alpha\dot{\psi}(x;t) - \psi''(x;t) + \int_x^{\infty} K_+(x,z;t)\left(\alpha\frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2}\right)\psi(z;t)dz\,. \tag{5.58}$$

Partially integrating we have

$$\int_x^{\infty} K_+\frac{\partial^2}{\partial z^2}\psi dz = -\widehat{K}_+\psi' + \frac{\partial}{\partial z}K_+\bigg|_{z=x}\psi + \int_x^{\infty}\frac{\partial^2}{\partial z^2}K_+\psi dz \tag{5.59}$$

where $\widehat{K}_+(x;t) = K_+(x,x;t)$. Setting $\Delta = \Delta_0 + \Delta_1$ we therefore get

$$0 = \Delta_1\left(\psi + \int_x^{\infty} K_+\psi dz\right) + \Delta_0\int_x^{\infty} K_+\psi dz - \alpha\int_x^{\infty} K_+\dot{\psi}dz - \widehat{K}_+\psi' + \frac{\partial}{\partial z}K_+\bigg|_{z=x}\psi + \int_x^{\infty}\frac{\partial^2}{\partial z^2}K_+\psi dz\,. \tag{5.60}$$

Now the second term is

$$\Delta_0 \int_x^\infty K_+ \psi dz = \alpha \int_x^\infty \dot{K}_+ \psi dz + \alpha \int_x^\infty K_+ \dot{\psi} dz + \frac{d}{dx}\widehat{K}_+ \psi + \widehat{K}_+ \psi' + K'_+\big|_{z=x} \psi - \int_x^\infty K''_+ \psi dz \tag{5.61}$$

and using this we get

$$0 = \Delta_1 \left(\psi + \int_x^\infty K_+ \psi dz\right) + \int_x^\infty \left(\alpha \dot{K}_+ - K''_+ + \frac{\partial^2}{\partial z^2} K_+\right) \psi dz + \frac{d}{dx}\widehat{K}_+ \psi + \left(K'_+ \frac{\partial}{\partial z} K_+\right)\Big|_{z=x} \psi \tag{5.62}$$

which simplifies to

$$0 = \left(\Delta_1 + 2\frac{d}{dx}\widehat{K}_+\right) \psi + \Delta_1 \int_x^\infty K_+ \psi dz + \int_x^\infty \left(\alpha \dot{K}_+ - K''_+ + \frac{\partial^2}{\partial z^2} K_+\right) \psi dz \,. \tag{5.63}$$

This must be true for all ψ which implies that

$$\Delta_1 = -2\frac{d}{dx}\widehat{K}_+ = -2\left(K'_+ \frac{\partial}{\partial z} K_+\right)\Big|_{z=x} \,, \tag{5.64}$$

so that Δ_1 is just a multiplicative operator and K_+ satisfies

$$\alpha \dot{K}_+ - K''_+ + \frac{\partial^2}{\partial z^2} K_+ + \Delta_1 K_+ = 0 \,. \tag{5.65}$$

These equations are similar to ones we found when discussing the KdV equation.

Returning now to the main development of the ZS scheme, the next step is to introduce a *pair* of operators Δ_0 and Δ. We take

$$\begin{aligned}
\Delta_0^{(1)} &= I\alpha \frac{\partial}{\partial t} - M_0 & \qquad \Delta_0^{(2)} &= I\beta \frac{\partial}{\partial y} + L_0 \\
\Delta^{(1)} &= I\alpha \frac{\partial}{\partial t} - M & \qquad \Delta^{(2)} &= I\beta \frac{\partial}{\partial y} + L
\end{aligned} \tag{5.66}$$

where α, β are constants and L_0, M_0, L, M are differential operators in x only. L_0, M_0 have constant coefficients and we require

$$0 = [\Delta_0^{(1)}, \Delta_0^{(2)}] = [L_0, M_0] \tag{5.67}$$

as well as

$$[\Delta_0^{(i)}, J_F] = 0 \qquad i = 1, 2 \,. \tag{5.68}$$

As before $\Delta^{(i)}$ is defined by

$$\Delta^{(i)}(I + J_+) = (I + J_+)\Delta_0^{(i)} \qquad i = 1, 2 \,. \tag{5.69}$$

Using this we find

$$[\Delta^{(1)}, \Delta^{(2)}](I + J_+) = \Delta^{(1)}\Delta^{(2)}(I + J_+) - (1 \leftrightarrow 2) = (I + J_+)[\Delta_0^{(1)}, \Delta_0^{(2)}] = 0 \tag{5.70}$$

and since $I + J_+$ is invertible this implies

$$[\Delta^{(1)}, \Delta^{(2)}] = 0 \,. \tag{5.71}$$

Using the expressions for these operators in (5.66) this becomes

$$0 = \left[I\alpha \frac{\partial}{\partial t} - M, I\beta \frac{\partial}{\partial y} + L \right] = \alpha \frac{\partial}{\partial t} L + \beta \frac{\partial}{\partial y} M + [L, M] \,. \tag{5.72}$$

This is a generalization of the Lax equation (5.14) to two auxiliary variables. Taking $\alpha = 1$, $\beta = 0$ recovers the original Lax equation.

Systems of non-linear evolution equations that can be formulated as

$$\alpha \frac{\partial}{\partial t} L + \beta \frac{\partial}{\partial y} M + [L, M] = 0 \tag{5.73}$$

can be solved by the ZS scheme. The procedure for solving them is as follows. The variable coefficients which arise in the 'dressed' operators L, M constitute the functions which satisfy the system of evolution equations. They are found from K_+, c.f. our example (5.64), where K_+ is a solution to the matrix Marchenko equation. This equation requires F and F is given by the solution to the pair of equations (5.68), c.f. our example (5.57). Note that in the ZS scheme the eigenvalue does not appear explicitly anywhere.

Before looking at how to carry out the solution of a system in detail, we will first look at some examples of equations that can be cast in the ZS form and therefore solve in this scheme.

5.3.3 Example 1: The KdV equation

In this case the operators involved are scalars. We set

$$\Delta_0^{(1)} = \frac{\partial}{\partial t} + 4 \frac{\partial^3}{\partial x^3} \qquad \Delta_0^{(2)} = -\frac{\partial^2}{\partial x^2} \,. \tag{5.74}$$

(i.e. $\alpha = 1$, $\beta = 0$, $M_0 = -\frac{\partial^3}{\partial x^3}$ and $L_0 = -\frac{\partial^2}{\partial x^2}$ in (5.66).) We also write

$$\Delta^{(2)} = L = -\frac{\partial^2}{\partial x^2} + u(x,t) \,. \tag{5.75}$$

Since $\Delta_0^{(2)}$ coincides with (5.54) for $\alpha = 0$ we conclude from (5.64) with $\Delta_1 = u$ that

$$u = -2 \frac{d}{dx} \widehat{K}_+ \tag{5.76}$$

and from (5.57) and (5.65) that

$$\frac{\partial^2}{\partial z^2} F - F'' = 0 \,, \qquad \frac{\partial^2}{\partial z^2} K_+ - K_+'' + u K_+ = 0 \,. \tag{5.77}$$

Similarly the condition $[\Delta_0^{(1)}, J_F] = 0$ turns out to give

$$\dot{F} + 4 \frac{\partial^3}{\partial z^3} F + 4F''' = 0 \,. \tag{5.78}$$

Finally we have

$$\Delta^{(1)} = \frac{\partial}{\partial t} - M \,, \qquad M = M_0 + M_1 = -4 \frac{\partial^3}{\partial x^3} + M_1 \,. \tag{5.79}$$

The condition $\Delta^{(1)}(I+J_+) = (I+J_+)\Delta_0^{(1)}$ becomes

$$-M_1\psi + \left(\frac{\partial}{\partial t} + 4\frac{\partial^3}{\partial x^3} - M_1\right)\int_x^\infty K_+\psi dz = \int_x^\infty K_+\left(\frac{\partial}{\partial t} + 4\frac{\partial^3}{\partial z^3}\right)\psi dz \tag{5.80}$$

or

$$-M_1\psi + \int_x^\infty \dot{K}_+\psi dz + 4\frac{\partial^3}{\partial x^3}\int_x^\infty K_+\psi dz - M_1\int_x^\infty K_+\psi dz = 4\int_x^\infty K_+\frac{\partial^3}{\partial z^3}\psi dz\,. \tag{5.81}$$

Expanding the $\frac{\partial^3}{\partial x^3}$-term and partially integrating the $\frac{\partial^3}{\partial z^3}$-term we get

$$\begin{aligned}&-M_1(\psi + \int_x^\infty K_+\psi dz) + \int_x^\infty \dot{K}_+\psi dz - 4\frac{\partial^2}{\partial x^2}(\widehat{K}_+\psi) - 4\frac{\partial}{\partial x}(K_+'|_{z=x}\psi) - 4K_+''|_{z=x}\psi\\ &+4\int_x^\infty K_+'''\psi dz = -4\widehat{K}_+\psi'' + 4\left.\frac{\partial}{\partial z}K_+\right|_{z=x}\psi' - 4\left.\frac{\partial^2}{\partial z^2}K_+\right|_{z=x}\psi - 4\int_x^\infty\frac{\partial^3}{\partial z^3}K_+\psi dz\end{aligned} \tag{5.82}$$

Now suppose that

$$M_1 = 4A(x,t)\frac{\partial}{\partial x} + 4B(x,t)\,. \tag{5.83}$$

The we get

$$\begin{aligned}&-4\left(\frac{d^2}{dx^2}\widehat{K}_+ + \frac{d}{dx}(K_+'|_{z=x}) + K_+''|_{z=x} - \left.\frac{\partial^2}{\partial z^2}K_+\right|_{z=x} - A\widehat{K}_+ + B\right)\psi\\ &-4\left(3\frac{d}{dx}\widehat{K}_+ + A\right)\psi' + \int_x^\infty\left(\dot{K}_+ + 4\frac{\partial^3}{\partial z^3}K_+ + 4K_+''' + 4AK_+' - 4BK_+\right)\psi dz = 0\end{aligned} \tag{5.84}$$

To cancel the ψ'-term we take $A = -3\frac{d}{dx}\widehat{K}_+ = \frac{3}{2}u$ and to cancel the first term

$$B = \left.\left(\frac{\partial^2}{\partial z^2}K_+ - K_+''\right)\right|_{z=x} - (K_+'' + \frac{\partial}{\partial z}K_+)|_{z=x} + \frac{3}{2}u\widehat{K}_+ - \frac{d^2}{dx^2}\widehat{K}_+ \tag{5.85}$$

which can be written as

$$B = \frac{3}{2}\left.\left(\frac{\partial^2}{\partial z^2}K_+ - K_+'' + uK_+\right)\right|_{z=x} - \frac{1}{2}(K_+'' + 2\frac{\partial}{\partial z}K_+ + \frac{\partial^2}{\partial z^2}K_+)|_{z=x} - \frac{d^2}{dx^2}\widehat{K}_+\,, \tag{5.86}$$

The first term vanishes by (5.77) and the second and third are the same so we get

$$B = -\frac{3}{2}\frac{d^2}{dx^2}\widehat{K}_+ = \frac{3}{4}u'\,. \tag{5.87}$$

And so we have

$$\Delta^{(1)} = \frac{\partial}{\partial t} + 4\frac{\partial^3}{\partial x^3} - 6u\frac{\partial}{\partial x} - 3u'\,. \tag{5.88}$$

The evolution equation

$$\alpha\frac{\partial}{\partial t}L + \beta\frac{\partial}{\partial y}M + [L,M] = 0 \tag{5.89}$$

becomes, setting $\alpha = 1$, $\beta = 0$, $L = -\frac{\partial^2}{\partial x^2} + u$ and $M = -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u'$,

$$0 = \dot{u} + [L,M] = \dot{u} - [\frac{\partial^2}{\partial x^2}, 6u\frac{\partial}{\partial x} + 3u'] - [-4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x}, u] = \dot{u} - 6uu' + u'''\,, \tag{5.90}$$

recovering the KdV equation. This shows how the ZS scheme can be applied to the KdV equation.

5.3.4 Example 2: The 2d KdV equation

The KdV case involved only one of the auxiliary variables, t. A straightforward generalization is to include also dependence on the other variable, y. We take

$$\Delta_0^{(1)} = \frac{\partial}{\partial t} + 4\frac{\partial^3}{\partial x^3} \qquad \Delta_0^{(2)} = \frac{\partial}{\partial y} - \frac{\partial^2}{\partial x^2}\,, \tag{5.91}$$

and write the dressed operators as

$$\Delta^{(2)} = \Delta_0^{(2)} + u(x,t,y)\,, \qquad \Delta^{(1)} = \Delta_0^{(1)} - 6u\frac{\partial}{\partial x} - 3u' + w(x,t,y)\,. \tag{5.92}$$

Following the same steps as in the previous section we find the equations for F

$$\frac{\partial}{\partial y}F + \frac{\partial^2}{\partial z^2}F - F'' = 0\,, \qquad \dot F + 4\frac{\partial^3}{\partial z^3}F + 4F''' = 0\,. \tag{5.93}$$

And in the end the equations for u and w become

$$\dot u - 6uu' + u''' - \frac{\partial}{\partial y}w = 0\,, \qquad w' = -3\frac{\partial}{\partial y}u\,. \tag{5.94}$$

Taking the x-derivative of the first equation and eliminating w using the second equation we find the equation for u

$$(\dot u - 6uu' + u''')' + 3\frac{\partial^2}{\partial y^2}u = 0\,. \tag{5.95}$$

This equation is known as the 2d KdV equation or the Kadomtsev-Petviashvili equation. It can also be solved by the ZS scheme.

So far we took the operators $\Delta_0^{(i)}$ to be scalars. It should be clear that much more freedom exists if we take them to be matrices. We will now consider such a case.

5.3.5 Example 3: The non-linear Schrödinger equation

Let us take

$$\Delta_0^{(1)} = I\left(i\alpha\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) \qquad \Delta_0^{(2)} = \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}\frac{\partial}{\partial x}\,, \tag{5.96}$$

where α, l, m are real constants and I is the 2×2 identity matrix. We have already seen that for an operator of the form of $\Delta_0^{(1)}$ above we have, see (5.54) and (5.64)

$$\Delta^{(1)} = \Delta_0^{(1)} + U(x,t) \tag{5.97}$$

where

$$U(x,t) = -2\frac{d}{dx}\widehat{K}_+(x,t)\,. \tag{5.98}$$

It remains to determine $\Delta^{(2)}$.

$$\Delta^{(2)} = \Delta_0^{(2)} + V(x,t) \tag{5.99}$$

the condition

$$\Delta^{(2)}(I + J_+) = (I + J_+)\Delta_0^{(2)} \tag{5.100}$$

gives

$$\Delta_0^{(2)} J_+\psi + V(\psi + J_+\psi) = J_+\Delta_0^{(2)}\psi \tag{5.101}$$

or

$$\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \frac{\partial}{\partial x}\int_x^\infty K_+\psi dz + V\left(\psi + \int_x^\infty K_+\psi dz\right) = \int_x^\infty K_+ \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \frac{\partial}{\partial z}\psi dz\,. \tag{5.102}$$

Partially integrating on the RHS and expanding the $\frac{\partial}{\partial x}$-term we get

$$\begin{aligned} &-\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}\widehat{K}_+\psi + \widehat{K}_+\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}\psi + V\psi \\ &\qquad + \int_x^\infty \left[\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} K_+' + \frac{\partial}{\partial z}K_+\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} + VK_+\right]\psi dz = 0\,. \end{aligned} \tag{5.103}$$

The unintegrated terms must vanish separately so that

$$V = \left[\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}, \widehat{K}_+\right]\,. \tag{5.104}$$

Writing

$$\widehat{K}_+ = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\,, \tag{5.105}$$

this becomes

$$V = (l-m)\begin{pmatrix} 0 & B \\ -C & 0 \end{pmatrix}\,. \tag{5.106}$$

From the vanishing of the integral terms we get the equation for K_+

$$\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} K_+' + \frac{\partial}{\partial z}K_+\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} + VK_+ = 0\,. \tag{5.107}$$

Evaluating this at $z = x$ we find for example

$$lA' = -(l-m)BC\,, \qquad mD' = (l-m)BC\,. \tag{5.108}$$

The conditions $[\Delta_0^{(i)}, J_F] = 0$ give the following equations for F

$$\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} F' + \frac{\partial}{\partial z}F\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} = 0\,, \qquad i\alpha\dot{F} + \frac{\partial^2}{\partial z^2}F - F'' = 0\,. \tag{5.109}$$

Now we let $B = u$ and $C = \pm u^*$, the complex conjugate of u. Then we have

$$\Delta^{(1)} = I\left(i\alpha\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) - 2\frac{d}{dx}\widehat{K}_+(x,t)\,, \tag{5.110}$$

where

$$-2\frac{d}{dx}\widehat{K}_+(x,t) = -2\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = -2\begin{pmatrix} \mp\frac{l-m}{l}|u|^2 & u' \\ \pm u'^* & \pm\frac{l-m}{m}|u|^2 \end{pmatrix}\,. \tag{5.111}$$

While

$$\Delta^{(2)} = \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \frac{\partial}{\partial x} + (l-m) \begin{pmatrix} 0 & u \\ \mp u^* & 0 \end{pmatrix} . \tag{5.112}$$

The Lax equation now becomes

$$\begin{aligned} 0 =& i\alpha(l-m) \begin{pmatrix} 0 & \dot{u} \\ \mp \dot{u}^* & 0 \end{pmatrix} \\ &+ \left[\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \frac{\partial}{\partial x} + (l-m) \begin{pmatrix} 0 & u \\ \mp u^* & 0 \end{pmatrix}, \frac{\partial^2}{\partial x^2} + 2 \begin{pmatrix} \mp \frac{l-m}{l}|u|^2 & u' \\ \pm u'^* & \pm \frac{l-m}{m}|u|^2 \end{pmatrix} \right] \end{aligned} \tag{5.113}$$

The commutator becomes

$$\begin{aligned} &2 \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \mp \frac{l-m}{l}(uu^*)' & u'' \\ \pm u''^* & \pm \frac{l-m}{m}(uu^*)' \end{pmatrix} + 2 \left[\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}, \begin{pmatrix} \mp \frac{l-m}{l}|u|^2 & u' \\ \pm u'^* & \pm \frac{l-m}{m}|u|^2 \end{pmatrix} \right] \frac{\partial}{\partial x} \\ &- 2(l-m) \begin{pmatrix} 0 & u' \\ \mp u'^* & 0 \end{pmatrix} \frac{\partial}{\partial x} - (l-m) \begin{pmatrix} 0 & u'' \\ \mp u''^* & 0 \end{pmatrix} \\ &+ 2(l-m) \left[\begin{pmatrix} 0 & u \\ \mp u^* & 0 \end{pmatrix}, \begin{pmatrix} \mp \frac{l-m}{l}|u|^2 & u' \\ \pm u'^* & \pm \frac{l-m}{m}|u|^2 \end{pmatrix} \right] . \end{aligned} \tag{5.114}$$

It is easy to see that the $\frac{\partial}{\partial x}$-terms cancel and we are left with

$$\begin{aligned} &2 \begin{pmatrix} \mp(l-m)(uu^*)' & lu'' \\ \pm m u''^* & \pm(l-m)(uu^*)' \end{pmatrix} - (l-m) \begin{pmatrix} 0 & u'' \\ \mp u''^* & 0 \end{pmatrix} \\ &+ 2(l-m) \begin{pmatrix} \pm(uu'^* + u^*u') & \pm \frac{l^2-m^2}{lm} u|u|^2 \\ \frac{l^2-m^2}{lm} u^*|u|^2 & \mp(u^*u' + u'^*u) \end{pmatrix} . \end{aligned} \tag{5.115}$$

The diagonal terms cancel and the Lax equation reduces to the equation

$$i\alpha(l-m)\dot{u} + (l+m)u'' \pm 2(l-m)\frac{l^2-m^2}{lm} u|u|^2 = 0 , \tag{5.116}$$

together with its complex conjugate. By rescaling x, t and u it can be brought to the form

$$i\dot{u} + u'' + u|u|^2 = 0 , \tag{5.117}$$

which is known as the non-linear Schrödinger (NLS) equation. This equation has applications to water waves, plasma waves and to the propagation of light in non-linear optical fibers. Since it can be cast in the Lax form it can be solved by the ZS scheme.

It is also possible to describe another important equation, the Sine-Gordon equation

$$u'' - \ddot{u} = \sin u \tag{5.118}$$

in the ZS scheme but we will not have to to do it here. Instead we will look at how solutions to the NLS equation can be constructed using the ZS scheme.

5.4 Solving the non-linear Schrödinger equation

We will see how to find the simplest solutions to the NLS equation

$$i\dot{u} + u'' + u|u|^2 = 0\,. \tag{5.119}$$

The first step is to solve the equations for F in (5.109). Writing

$$F = \begin{pmatrix} 0 & r \\ s & 0 \end{pmatrix} \tag{5.120}$$

the first equation becomes

$$\begin{pmatrix} 0 & lr' \\ ms' & 0 \end{pmatrix} + \begin{pmatrix} 0 & m\frac{\partial}{\partial z}r \\ l\frac{\partial}{\partial z}s & 0 \end{pmatrix} = 0\,, \tag{5.121}$$

or

$$lr' + m\frac{\partial}{\partial z}r = 0\,, \qquad ms' + l\frac{\partial}{\partial z}s = 0\,, \tag{5.122}$$

which implies

$$r = r(mx - lz;t)\,, \qquad s = s(lx - mz;t)\,. \tag{5.123}$$

Using this the second equation for F becomes

$$i\alpha\begin{pmatrix} 0 & \dot{r} \\ \dot{s} & 0 \end{pmatrix} + \begin{pmatrix} 0 & (l^2-m^2)r'' \\ (m^2-l^2)s'' & 0 \end{pmatrix} = 0\,, \tag{5.124}$$

where the prime denotes the derivative with respect to the first variable. It is clear that there exists an exponential solution

$$r(x,z;t) = r_0 e^{\rho(mx-lz)+i\rho^2(l^2-m^2)t/\alpha}\,, \qquad s(x,z;t) = s_0 e^{\sigma(lx-mz)+i\sigma^2(m^2-l^2)t/\alpha}\,, \tag{5.125}$$

where r_0, s_0, ρ, σ are arbitrary constants.

The next step is to plug the solution for F into the Marchenko equation for K_+

$$K_+(x,z,;t) + F(x,z;t) + \int_x^\infty K_+(x,y;t)F(y,z;t)dy = 0\,. \tag{5.126}$$

Writing

$$K_+ = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.127}$$

this becomes, suppressing the arguments,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & r \\ s & 0 \end{pmatrix} + \int_x^\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & r \\ s & 0 \end{pmatrix} dy = 0\,, \tag{5.128}$$

so that

$$a(x,z;t) + \int_x^\infty b(x,y;t)s(y,z;t)dy = 0 \tag{5.129}$$

$$b(x,z;t) + r(x,z;t) + \int_x^\infty a(x,y;t)r(y,z;t)dy = 0 \tag{5.130}$$

and two similar equations for c and d. Looking at the z-dependence we find that

$$a(x,z;t) = e^{-\sigma m z} L(x,t)\,, \qquad b(x,z;t) = e^{-\rho l z} M(x,t)\,. \tag{5.131}$$

Using this the equations reduce to

$$L + s_0 M \int_x^\infty e^{(\sigma-\rho)ly + i\sigma^2(m^2-l^2)t/\alpha} dy = 0 \tag{5.132}$$

$$M + r_0 e^{\rho m x + i\rho^2(l^2-m^2)t/\alpha} + r_0 L \int_x^\infty e^{(\rho-\sigma)my + i\rho^2(l^2-m^2)t/\alpha} dy = 0\,. \tag{5.133}$$

For the integrals to be well defined we need

$$\Re((\sigma-\rho)l) < 0 \qquad \text{and} \qquad \Re((\rho-\sigma)m) < 0\,, \tag{5.134}$$

so that l and m must have opposite sign. Let us take

$$l = 2\,, \qquad m = -1 \qquad \text{and} \qquad \alpha = \frac{1}{3}\,. \tag{5.135}$$

The NLS equation (5.116) for the lower sign becomes

$$i\dot{u} + u'' + 9u|u|^2 = 0\,, \tag{5.136}$$

so at the end we will need to take $u \to u/3$ to get a solution to the equation in the standard form (5.119). Provided that $\Re(\sigma-\rho) < 0$ we get the following equation for M (solving the first equation for L and plugging into the second)

$$M\left(1 - r_0 s_0 e^{9i(\rho^2-\sigma^2)t} \int_x^\infty e^{2(\sigma-\rho)y} dy \int_x^\infty e^{(\sigma-\rho)y} dy\right) = -r_0 e^{-\rho x + 9i\rho^2 t}\,. \tag{5.137}$$

Doing the integrals this reduces to

$$M\left(1 - \frac{r_0 s_0}{2(\rho-\sigma)^2} e^{3(\sigma-\rho)x + 9i(\rho^2-\sigma^2)t}\right) = -r_0 e^{-\rho x + 9i\rho^2 t}\,. \tag{5.138}$$

Setting

$$\rho = k + i\lambda\,, \qquad \sigma = -k + i\lambda\,, \qquad \frac{r_0 s_0}{8k^2} = -1\,, \tag{5.139}$$

with $k > 0$ we get

$$M = -r_0 \frac{e^{-(k+i\lambda)x + 9i(k+i\lambda)^2 t}}{e^{-6kx - 36k\lambda t} + 1}\,. \tag{5.140}$$

Therefore

$$b = e^{-2\rho z} M = -r_0 \frac{e^{-(k+i\lambda)(x+2z) + 9i(k+i\lambda)^2 t}}{e^{-6kx - 36k\lambda t} + 1}\,. \tag{5.141}$$

But $u(x,t) = B(x,t) = b(x,x;t)$, see (5.105), so we get

$$\begin{aligned} u &= -\, r_0 \frac{e^{-3(k+i\lambda)x + 9i(k+i\lambda)^2 t}}{e^{-6kx-36k\lambda t} + 1} = -r_0 e^{-3i\lambda x + 9i(k^2-\lambda^2)t} \frac{e^{-3kx - 18k\lambda t}}{e^{-6kx - 36k\lambda t} + 1} \\ &= -\, \frac{r_0}{2} e^{-3i\lambda x + 9i(k^2-\lambda^2)t} \operatorname{sech}\left(3kx + 18k\lambda t\right)\,. \end{aligned} \tag{5.142}$$

Finally the equations for c and d and the requirement that $C(x;t) = c(x,x;t) = -u^*$ implies $s_0 = -r_0$ so that $r_0 s_0 = -8k^2$ gives $r_0 = \pm 2\sqrt{2}k$. To get a solution to our original equation (5.119) we also need to send $u \to u/3$ and doing this we find the solitary wave solution to the NLS equation

$$u(x,t) = \pm a e^{i\left[\frac{c}{2}(x-ct)+nt\right]} \operatorname{sech}\left(\frac{a}{\sqrt{2}}(x-ct)\right) , \tag{5.143}$$

where

$$a = 3\sqrt{2}k\,, \qquad c = -6\lambda \qquad \text{and} \qquad n = \frac{1}{2}a^2 + \frac{1}{4}c^2\,. \tag{5.144}$$

The generalization to the N-soliton solution is now straight-forward. Instead of constructing r, s in F from a single exponential as we did one now takes a sum of N of them.

Unfortunately we have to leave the subject of the ZS scheme here, although there is much more to be said. Hopefully this course has served as an invitation to the vast and fascinating subject of integrable models which will inspire you to learn more.