Exam Problems: Non-linear waves and solitons, Spring 2020

1. *Two-soliton solution.* Construct the two-soliton solution of the non-linear Schrödinger equation

$$i\dot{u} + u'' + u|u|^2 = 0 \,.$$

2. *Zero curvature representation and conservation laws.* Consider the linear system of equations

$$V' = LV \,, \qquad \dot{V} = MV \,,$$

where

$$L = \frac{i}{2}\begin{pmatrix} \lambda & u^* \\ 2u & -\lambda \end{pmatrix} \,, \quad M = \frac{i}{2}\begin{pmatrix} \lambda^2 - |u|^2 & -iu'^* + \lambda u^* \\ 2iu' + 2\lambda u & -\lambda^2 + |u|^2 \end{pmatrix}$$

and V is a two-component vector $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$. Here L, M, V all depend on x, t as well as the auxiliary spectral parameter λ. Show that the compatibility condition for the linear problem leads to the non-linear Schrödinger equation for u.

Define the functions ρ_k recursively as

$$\rho_k = -\frac{u'}{u}\rho_{k-1} + \rho'_{k-1} + \tfrac{1}{2}\sum_{j=0}^{k-2}\rho_j\rho_{k-2-j} \,, \qquad \rho_0 = |u|^2 \,.$$

It turns out that the charges

$$Q_k = \int_{-\infty}^{\infty} \rho_k dx$$

are conserved. Show this for $k = 0, 1, 2$.

3. *Sine-Gordon equation.* Show that the Sine-Gordon (SG) equation

$$u'' - \ddot{u} = \sin u$$

can be cast in the ZS scheme by taking the 4×4 matrix operators

$$\Delta_0^{(1)} = \alpha I \frac{\partial}{\partial t} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\frac{\partial}{\partial x} \,, \qquad \Delta_0^{(2)} = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & 0 \end{pmatrix}\frac{\partial}{\partial x} \,,$$

where I denotes the 4×4 and 2×2 unit matrices (in the block matrices 0 denotes the 2×2 zero matrix) and $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is one of the Pauli matrices. You can assume that

$$\widehat{K} = \begin{pmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & \widehat{D} \end{pmatrix}$$

satisfies $[\sigma_2, \widehat{A}] = w\sigma_1$, $\widehat{B} = -i\sigma_2\tilde{D}$ and $\widehat{C} = i\tilde{D}\sigma_2$ where

$$\tilde{D} = \frac{1}{4}\begin{pmatrix} e^{iu/2} & 0 \\ 0 & e^{-iu/2} \end{pmatrix}$$

and u, w are functions of x, t.