Problems Week 5

1. *A* sech^2 *potential.* Verify that the solution to the associated Legendre equation

$$\hat{\psi}(x;k) = a(k)2^{ik}(\operatorname{sech} x)^{-ik} F(c_+, c_-; 1-ik; \frac{1}{2}(1+\tanh x))\,,$$

where

$$c_\pm = \frac{1}{2} - ik \pm \sqrt{U_0 + \frac{1}{4}}$$

behaves asymptotically as

$$\hat{\psi} = \begin{cases} e^{-ikx} + be^{ikx} & x \to +\infty \\ ae^{-ikx} & x \to -\infty \end{cases}$$

for

$$a(k) = \frac{\Gamma(c_+)\Gamma(c_-)}{\Gamma(1-ik)\Gamma(ik)}\,, \qquad b(k) = a(k)\frac{\Gamma(1-ik)\Gamma(-ik)}{\Gamma(c_+ + ik)\Gamma(c_- + ik)}\,.$$

The hypergeometric function is defined as

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b_n)}{(c)_n} \frac{z^n}{n!}\,, \qquad (q)_n = q(q{+}1)\cdots(q{+}n{-}1) \qquad (q)_0 = 1\,.$$

It satisfies the following identity due to Euler

$$\begin{aligned} F(a,b;c;z) = &\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b+1-c;1-z) \\ &+ (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a,c-b;1+c-a-b;1-z)\,. \end{aligned}$$

2. *The scattering coefficients.* With $u(x) = -U_0 \operatorname{sech}^2 x$ show, by using the properties of the gamma function, that $a(k)$ and $b(k)$ in the previous problem satisfy the conditions

$$|a|^2 + |b|^2 = 1$$

and $a \to 1$ and $b \to 0$ as $k \to \infty$.

3. *Inverse scattering about* $-\infty$. Find the equation for $L(x,z)$ if

$$\psi_-(x;k) = e^{-ikx} + \int_{-\infty}^{x} L(x,z)e^{-ikz}\,dz$$

is a solution of $\psi'' + (\lambda - u(x))\psi = 0$. What boundary conditions must $L(x,z)$ satisfy?