integral parts of the definition. In later chapters, where some generalisations of the inverse scattering method will be introduced, the reader may test the usefulness of our definition for the solutions of other evolution equations.

Further reading

Other accounts of the connection between the KdV equation and inverse scattering are by Miura (1976); Ablowitz & Segur (1981, Chap. 2); Dodd, Eilbeck, Gibbon & Morris, (1982, Chap. 3); Newell (1985, Chap. 1). The long-time behaviour of the solutions is discussed by Ablowitz & Segur (1981, §1.7).

Much of this work on the KdV equation was initiated by the publication of the seminal papers of Gardner, Greene, Kruskal & Miura (1967, 1974).

Exercises

*Q4.1 *Alternative derivation.*

(i) Show that, if $K(x, z)$ satisfies the Marchenko equation

$$K(x,z) + F(x,z) + \int_x^\infty K(x,y)F(y,z)\,\mathrm{d}y = 0,$$

where F is a solution of

$$F_{xx} - F_{zz} = 0,$$

then

$$K_{xx} - K_{zz} - uK = 0, \tag{1}$$

where $u(x) = -2(\mathrm{d}/\mathrm{d}x)K(x,x)$ and K, $K_z \to 0$ as $z \to +\infty$.

(ii) Now suppose that $F = F(x, z; t)$ and $K = K(x, z; t)$, with

$$F_t + 4(F_{xxx} + F_{zzz}) = 0,$$

and show that

$$K_t + 4(K_{xxx} + K_{zzz}) - 3u_x K - 6uK_x = 0. \tag{2}$$

(iii) Use (1) and (2) to show that

$$K_t + \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\right)^3 K - 3u(K_x + K_z) = 0$$

and hence that

$$u_t - 6uu_x + u_{xxx} = 0.$$

*Q4.2 *Concentric KdV equation.*

(i) Show that, if $K(x, z; t)$ satisfies the Marchenko equation

$$K(x,z;t) + F(x,z;t) + \int_x^\infty K(x,y;t)F(y,z;t)\,\mathrm{d}y = 0,$$