# LINEAR ALGEBRA - WEEK 4
## APPENDIX

## VECTOR SPACES WITH SCALAR PRODUCT

**Definition** Let $U$ be a vector space over $\mathbb{R}$. A map $\langle \, , \, \rangle : U \times U \longrightarrow \mathbb{R}$ is called **scalar product** on $U$ if it is symmetric billinear form with corresponding quadratic form positive definite, i.e. it means that

① $\forall u, v, w \in U, \forall a, b \in \mathbb{R}$

$$\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$$

$$\langle u, av + bw \rangle = a\langle u, v \rangle + b\langle u, w \rangle$$

② $\langle u, v \rangle = \langle v, u \rangle$

③ $\langle u, u \rangle > 0$ for $u \neq \vec{0}$.

**Example** ① Standard scalar product on $\mathbb{R}^n$

$U = \mathbb{R}^n \qquad \langle x, y \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

**Example** ② $C[a,b]$ is the vector space of continuous real functions on the interval $[a,b]$

$\langle \, , \, \rangle : C[a,b] \times C[a,b] \longrightarrow \mathbb{R}$

$$\langle f, g\rangle = \int_a^b f(x)\, g(x)\, dx .$$

<u>Definition</u> Let $\mathcal{U}$ be a vector space over $\mathbb{C}$ (complex numbers). A map

$$\langle \, , \, \rangle : \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{C}$$

is called <u>a scalar product</u> on $\mathcal{U}$ if

(1) $\forall u, v, w \in \mathcal{U}$, $a, b \in \mathbb{C}$

$$\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$$

(2) $\langle u, av + bw\rangle = \bar{a}\langle u, v\rangle + \bar{b}\langle u, w\rangle$

(3) $\langle u, v\rangle = \overline{\langle u, v\rangle}$

(4) $\langle u, u\rangle > 0$ for $u \neq \vec{0}$

Here the bar $\bar{\ }$ means complex conjugation

$$a = a_1 + i a_2, \quad a_1, a_2 \in \mathbb{R}$$

$$\bar{a} = a_1 - i a_2$$

$$a \cdot \bar{a} = a_1^2 + a_2^2 = |a|^2 .$$

<u>Example</u> 1 Standard scalar product on $\mathbb{C}^n$: $\langle \, , \, \rangle : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$

$$\langle x, y\rangle = x_1 \bar{y}_1 + x_2 \bar{y}_2 + \dots + x_n \bar{y}_n$$

$$\langle x, x\rangle = x_1 \bar{x}_1 + x_2 \bar{x}_2 + \dots + x_n \bar{x}_n = |x_1|^2 + |x_2|^2 + \dots |x_n|^2$$

<u>Example 2</u> $C([a,b], \mathbb{C})$ the vector space of continuous complex functions on the intervae $[a,b]$.

$$\langle f, g \rangle = \int_a^b f(x)\,\overline{g(x)}\,dx .$$

<u>The norm</u> of the vector $u \in U$ is

$$\|u\| = \sqrt{\langle u, u \rangle} .$$

<u>Example</u> Consider $\mathbb{R}^4$ with standard scalar product. Then

$$\|(1, 2, 3, -2)\| = \sqrt{1^2 + 2^2 + 3^2 + (-2)^2} = \sqrt{18}$$

Two vectors $u, v \in U$ are orthogonal, if $\langle u, v \rangle = 0.$

We write $u \perp v$.

<u>CAUCHY ENEQUALITY</u>

Let $u, v \in U$. Then

$$|\langle u, v \rangle| \le \|u\| \cdot \|v\|$$

The equality is just for linearly dependent vectors $u$ and $v$.

**Example** Consider $U = \mathbb{R}^n$ with standard scalar product. Then

$$\underbrace{|x_1 y_1 + x_2 y_2 + \dots + x_n y_n|}_{= |\langle x, y \rangle|} \le \underbrace{\sqrt{x_1^2 + \dots + x_n^2}\sqrt{y_1^2 + \dots y_n^2}}_{= \|x\| \cdot \|y\|}$$

**Proof for real space $U$**

If $v = \vec{0}$ then

$$|\langle u, \vec{0} \rangle| = 0 = \|\vec{0}\| \|u\|$$

and theorem holds. If $v \neq \vec{0}$, consider the vectors $tv - u$ for all $t \in \mathbb{R}$.

$$0 \le \|tv - u\|^2 = \langle tv - u, tv - u \rangle =$$

$$= t^2 \langle v, v \rangle - 2t \langle u, v \rangle + \|u\|^2$$

This is quadratic polynomial in $t$ which is $\ge 0$. That is why its discriminant $D$ is less or equal to $0$.

$$D = 4 \langle u, v \rangle^2 - 4 \langle v, v \rangle \langle u, u \rangle$$

$$= 4 \langle u, v \rangle^2 - 4 \|v\|^2 \|u\|^2 \le 0$$

Hence

$$\langle u, v \rangle^2 \le \|u\|^2 \|v\|^2$$

$$|\langle u, v \rangle| \le \|u\| \|v\|.$$

The equality occurs only if $tv - u = \vec{0}$.