LA – WEEK 7: EIGENVALUES AND EIGENVECTORS

<u>Linear operator</u> (or linear endomorphism or linear transformation) is a linear map

$$\varphi : U \to U,$$

where U is a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Always it holds $\varphi(U) \subseteq U$, $\varphi(\{\vec{0}\}) = \{\vec{0}\}$.
U and $\{\vec{0}\}$ are so called trivial invariant subspaces.

The vector subspace $V \subseteq U$ is called <u>invariant subspace</u> of an operator $\varphi : U \to U$, if

$$\varphi(V) \subseteq V.$$

<u>Example 1</u> $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ $\quad \varphi(x) = Ax$

$$A = \begin{pmatrix} 1 & -1 & 1 & -3 \\ 2 & 1 & 0 & 2 \\ -2 & 3 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}$$

Consider the subspace

$$V = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right] = [u_1, u_2]$$

We show that V is an invariant subspace for φ.

$$\varphi(u_1) = \begin{pmatrix} 1 \\ 2 \\ -2 \\ 0 \end{pmatrix} = u_1 + 2u_2 \in V \qquad \varphi(u_2) = \begin{pmatrix} -2 \\ 1 \\ -1 \\ 0 \end{pmatrix} = -2u_1 + u_2 \in V$$

Hence $\varphi(au_1 + bu_2) = a\,\underset{\overset{\Uparrow}{V}}{\varphi(u_1)} + b\,\underset{\overset{\Uparrow}{V}}{\varphi(u_2)} \in V.$

<u>Definition</u> The matrix of the operator $\varphi : U \to U$ in the basis $\alpha = (u_1, u_2, \dots u_n)$ of the vector space U is a matrix whose columns are coordinates of vectors $\varphi(u_1), \varphi(u_2), \dots, \varphi(u_n)$ in the basis α.

We write

$$(\varphi)_{\alpha,\alpha} = \Big((\varphi(u_1))_\alpha \ \ (\varphi(u_2))_\alpha \cdots (\varphi(u_n))_\alpha \Big)$$

coordinates of the vector $\varphi(u_1)$ in the basis α.

<u>Example</u> Let us return to the example 1.

For the basis $\varepsilon = \left(e_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, e_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, e_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right)$

we have $$(\varphi)_{\varepsilon,\varepsilon} = A.$$

For the basis $\alpha = \left(u_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, u_2 = \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}, e_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, e_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right)$

we have

$\varphi(u_1) = u_1 + 2u_2 = u_1 + 2u_2 + 0\cdot e_3 + 0\cdot e_4$

$\varphi(u_2) = -2u_1 + u_2 = -2u_1 + 1\cdot u_2 + 0\cdot e_3 + 0\cdot e_4$

$\varphi(e_3) = u_1 + 4e_3 - e_4 = 1\cdot u_1 + 0\cdot u_2 + 4e_3 - 1e_4$

$\varphi(e_4) = -3u_1 + 2u_2 + e_3 + 4e_4$

That is why

$$(\varphi)_{\alpha,\alpha} = \begin{pmatrix} 1 & -2 & 1 & -3 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & -1 & 4 \end{pmatrix}$$

Example 1 - prolongation

Consider another subspace

$$W = \left[\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right] = [u_3, u_4].$$

$\varphi(u_3) = 4u_3 - u_4 \in W$

$\varphi(u_4) = u_3 + 4u_4 \in W$

Hence W is also an invariant subspace for φ.

Consider the basis $\mathcal{B} = (u_1, u_2, u_3, u_4)$.

In this basis

$$(\varphi)_{\mathcal{B},\mathcal{B}} = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & -1 & 4 \end{pmatrix}.$$

We have $\mathbb{R}^4 = V \oplus W$.

1-dim invariant subspaces - eigenvectors

Let $\varphi : U \to U$ be linear operator and $u \in U \setminus \{\vec{o}\}$.

If the one-dimensional space $[u] \subset U$ is invariant then

$$\varphi(u) \in [u]$$

i.e. $\exists \lambda \in \mathbb{K}$

$$\varphi(u) = \lambda u.$$

Then for all multiples of u we have

$$\varphi(ku) = k\varphi(u) = k\lambda u = \lambda(ku).$$

So on every one-dimensional invariant subspace the operator φ is a multiplication by $\lambda \in \mathbb{K}$.

<u>Definition</u> <u>Nonzero</u> vector $u \in U$ is called eigenvector for $\varphi: U \to U$ if there is a $\lambda \in \mathbb{K}$ such that

$$\varphi(u) = \lambda u.$$

λ is called an <u>eigenvalue</u> corresponding to this eigenvector.

<u>Computation of eigenvectors</u>

Let first $U = \mathbb{K}^n$ and $\varphi(x) = Ax$ where A is a matrix n by n.

Then the existence of $x \neq 0$ (and $\lambda \in \mathbb{K}$) such that

$$\varphi(x) = \lambda x$$

is equivalent to

- $\exists \lambda \quad \exists x \neq 0 \qquad Ax = \lambda x$
- $\exists \lambda \quad \exists x \neq 0 \qquad Ax - \lambda x = 0$
- $\exists \lambda \quad \exists x \neq 0 \qquad (A - \lambda E)x = 0 \qquad E = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix}$
- $\exists \lambda \qquad \det(A - \lambda E) = 0$

<u>Characteristic polynomial</u> of a matrix A is

$$p(\lambda) = \det(A - \lambda E) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots \\ a_{21} & a_{22} - \lambda & \cdots \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots \end{pmatrix}$$

$$= (-\lambda)^n + b_1 \lambda^{n-1} + \cdots + b_1 \lambda + b_0$$

<u>Lemma</u> $\lambda_0 \in \mathbb{K}$ is an eigenvalue of the operator φ, $\varphi(x) = Ax$, if and only if λ_0 is a root of ~~[illegible]~~ the characteristic polynomial $\det(A - \lambda E)$.

If we know the eigenvalue $\lambda_0 \in \mathbb{K}$ then we can determine corresponding eigevectors by solving the system of homogeneous linear equations

$$(A - \lambda_0 E)x = 0.$$

<u>Computation of eigenvalues for general linear operators</u> $\varphi : U \to U$.

We take a basis α of U. In this basis $(\varphi)_{\alpha\alpha} = A$ and we have an operator

$$\mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad x \longmapsto Ax.$$

The eigenvalues of A are eigenvalues of φ. The eigenvectors of A are coordinates of eigenvector ~~[illegible]~~ of φ in the basis α.

If α and β are two different basis of U then we have:

$$(\varphi)_{\beta,\beta} = (id)_{\beta,\alpha} \cdot (\varphi)_{\alpha,\alpha} \cdot (id)_{\alpha,\beta}$$

where $(id)_{\beta,\alpha} = (id)^{-1}_{\alpha,\beta}$.

If $\alpha = (u_1, \dots u_n)$, $\beta = (v_1, v_2, \dots v_n)$

then $(id)_{\alpha,\beta} = \left((v_1)_\alpha, (v_2)_\alpha, \dots, (v_n)_\alpha \right)$.

Matrices A and B are called <u>similar</u> if there is a matrix P regular, such that

$$B = P^{-1} A P.$$

(Matrices $(\varphi)_{\alpha,\alpha}$ and $(\varphi)_{\beta,\beta}$ are similar.)

<u>Lemma</u> Similar matrices have the same characteristic polynomial.

<u>Proof</u>: Let $B = P^{-1} A P$. Then

$$\det (B - \lambda E) = \det \left(P^{-1} A P - \lambda P^{-1} E P \right) =$$

$$= \det P^{-1} (A - \lambda E) P = \det P^{-1} . \det (A - \lambda E) \cdot \det P = \frac{1}{\det P} \cdot \det (A - \lambda E) \cdot \det P =$$

$$= \det (A - \lambda E).$$

<u>Characteristic polynomial</u> of an operator $\varphi : U \to U$ is the characteristic polynomial of any matrix $(\varphi)_{\alpha,\alpha}$ where α is any basis of U. (All such polynomials are the same.)

<u>Example 2</u> Find eigenvalues and eigenvectors of the operator $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$, $\varphi(x) = Ax$,

$$A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 3 & 0 \\ 2 & -2 & 2 \end{pmatrix}$$

$$\text{del} \begin{pmatrix} -1-\lambda & 1 & 0 \\ -1 & 3-\lambda & 0 \\ 2 & -2 & 2-\lambda \end{pmatrix} = \lambda^3 - 4\lambda^3 + 2\lambda + 4$$

The roots are $2, 1+\sqrt{3}, 1-\sqrt{3}$.

Eigenvectors to the eigenvalue 2 are

$$(0,0,p) \qquad p \neq 0$$

Compute the eigenvectors for $1+\sqrt{3}$ and $1-\sqrt{3}$.

<u>HOMEWORK 7 ~~Example 2~~</u> Find eigenvalues and eigenvectors of the operator $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$, $\varphi(x) = Ax$,

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 3 \end{pmatrix}.$$