WEEK 9+10 SINGULAR DECOMPOSITION OF MATRICES

Notation If A is a complex matrix we will write
$$A^* = \bar{A}^T.$$
For A a real matrix we put
$$A^* = A^T$$
A is hermitian (symmetric) if $A = A^*$.

Example: $A = \begin{pmatrix} 2+i & -3 \\ 1-i & 4+2i \\ i & 6+3i \end{pmatrix}$

$$A^* = \begin{pmatrix} 2-i & 1+i & -i \\ -3 & 4-2i & 6-3i \end{pmatrix}$$

Lemma Let A be a matrix $k \times n$. Then $AA^* \in Mat_{k\times k}(\mathbb{K})$ and $A^*A \in Mat_{n\times n}(\mathbb{K})$ are hermitian matrices a their eigenvalues are ≥ 0.

Proof: $(AA^*) = (A^*)^* A^* = AA^*$

$\langle AA^*x, x\rangle = \langle A^*x, A^*x\rangle \geq 0.$

Lemma $\ker A^*A = \ker A$

Proof: If $x \in \ker A$, then $A^*Ax = A^*0 = 0$, i.e. $x \in \ker A^*A$.

If $x \in \ker A^*A$, then $A^*Ax = 0$

$$\begin{aligned} \langle A^*Ax, x\rangle &= 0 \\ \langle Ax, Ax\rangle &= 0 \\ \|Ax\| &= 0 \\ Ax &= 0 \end{aligned}$$

i.e. $x \in \ker A$. ▨

Theorem on singular decomposition

Let $A \in \mathrm{Mat}_{k\times n}(\mathbb{K})$. Then there are unitary (orthogonal) matrices $P \in \mathrm{Mat}_{k\times k}(\mathbb{K})$ and $Q \in \mathrm{Mat}_{n\times n}(\mathbb{K})$ such that

$$A = PSQ^*$$

where

$$S = \left(\begin{array}{ccc|c} s_1 & & 0 & \\ & s_2 & & 0 \\ 0 & & \ddots & \\ & & s_r & \\ \hline & 0 & & 0 \end{array}\right) \in \mathrm{Mat}_{k\times n}(\mathbb{R})$$

and $s_1, \dots, s_r$ are $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots \sqrt{\lambda_r}$ of positive eigenvalues of the matrix A^*A.

<u>Proof</u> gives the way how to find this decomposition.

Consider $\varphi : \mathbb{K}^n \longrightarrow \mathbb{K}^k$, $\varphi(x) = Ax$.
Then $\varphi^*\varphi : \mathbb{K}^n \longrightarrow \mathbb{K}^n$, $\varphi^*\varphi(x) = A^*Ax$ is a selfadjoint operator with nonnegative eigenvalues. There is an orthonormal basis $\alpha = (u_1, \dots u_r, u_{r+1}, \dots u_n)$ in K^n formed from eigenvectors with eigenvalues

$$\lambda_1, \lambda_2, \dots \lambda_r > 0 \text{ and } \lambda_{r+1} = \dots = \lambda_n = 0$$

Vectors $\varphi(u_1), \varphi(u_2), \dots \varphi(u_r)$ are mutually orthogonal and $\|\varphi(u_i)\| = \sqrt{\lambda_i}$.
Put $v_i = \dfrac{\varphi(u_i)}{\sqrt{\lambda_i}}$ $\quad 1 \le i \le r$

and complete vector $v_1, v_2, \dots v_r$ into an ortonormal basis β of $\mathbb{K}^k$. Then

$$(\varphi)_{\beta,\alpha} = \begin{array}{c} \left(\begin{array}{ccc|c} s_1 & \cdots & 0 & 0 \\ 0 & & s_r & \\ \hline & 0 & & 0 \end{array}\right)\begin{array}{l} \}\, r \\ \\ \}\, k-r \end{array} \\ \underbrace{\qquad\quad}_{r}\;\underbrace{\quad}_{n-r} \end{array} \in \mathrm{Mat}_{k \times n}(\mathbb{K})$$

So we have

$$A = (\varphi)_{\varepsilon_k, \varepsilon_n} = (id)_{\varepsilon_k, \beta}\,(\varphi)_{\beta,\alpha}\,(id)_{\alpha, \varepsilon_n}$$
$$= P\,S\,Q^*$$

Algorithm for finding singular decomposition.

Find eigenvalues and eigenvectors of $A^* A$. The ortonormal basis formed by eigenvalues gives the matrix $Q = (u_1\ u_2, \dots u_n)$.

The ~~images~~ vectors $Au_1, Au_2, \dots Au_r$ are $\neq \vec{0}$, complete them into ortonormal basis

$$v_1 = \frac{Au}{\sqrt{\lambda_1}}, \quad v_2 = \frac{Au}{\sqrt{\lambda_2}}, \dots, v_r = \frac{Au}{\sqrt{\lambda_r}}, \quad v_{r+1}, \dots v_k$$

Then $P = (v_1, v_2, \dots v_k)$.

$$A = P B Q^*.$$

Example $\quad A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$A^* A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} \qquad \det \begin{pmatrix} 2-\lambda & 2 \\ 2 & 5-\lambda \end{pmatrix} = \lambda^2 - 7\lambda + 6$$

$$\lambda_{1,2} = 1, 6 \qquad u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad u_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$Q = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$v_1 = \frac{Au_1}{\sqrt{1}} = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

$$v_2 = \frac{Au_2}{\sqrt{6}} = \frac{1}{\sqrt{30}} \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$$

$$v_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 5/\sqrt{30} & -1/\sqrt{6} \\ 2/\sqrt{5} & 1/\sqrt{30} & 1/\sqrt{6} \\ -1/\sqrt{5} & 2/\sqrt{30} & 2/\sqrt{6} \end{pmatrix}$$

$$A = P \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{6} \\ 0 & 0 \end{pmatrix} Q^*$$

PSEUDOINVERSE MATRIX

Inverse matrix to the matrix $A \in Mat_{n \times n}(\mathbb{K})$ is A^{-1} such that

$$A A^{-1} = E \quad \text{(unit matrix)}$$
$$A^{-1} A = E$$

If we have singular decomposition of A

$$A = P S Q^*$$

where

$$S = \begin{pmatrix} s_1 & & & 0 \\ & s_2 & & \\ & & \ddots & \\ 0 & & & s_n \end{pmatrix} \quad \text{where } s_i > 0.$$

Then

$$A^{-1} = (P S Q^*)^{-1} =$$
$$= (Q^*)^{-1} S^{-1} P^{-1} =$$
$$= Q S^{-1} P^* .$$

Using this observation we can generalize the definition of inverse matrix.

<u>Def</u> If $A \in Mat_{k \times n}(\mathbb{K})$, then the pseudoinverse $A^{(-1)}$ to A is

$$A^{(-1)} = \underbrace{Q}_{n \times n} \underbrace{\begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix}}_{n \times k} \underbrace{P^*}_{k \times k} \in Mat_{n \times k}(\mathbb{K})$$

7

if
$$A = \underbrace{P}_{k\times k} \underbrace{\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}}_{k\times n} \underbrace{Q^*}_{n\times n}$$

Properties of pseudoinverses

1. If A is invertible, then
$$A^{(-1)} = A^{-1}$$

~~2 [illegible]~~

2. If $x \in K^n$ then $A^{(-1)} A x$ is an ortogonal projection of x into $\ker A$.

3. $$\left(A^{(-1)}\right)^{(-1)} = A.$$

4. $A^{(-1)} A$ and $A A^{(-1)}$ are selfadjoint.

5. It holds
$$A A^{(-1)} A = A$$
$$A^{(-1)} A A^{(-1)} = A^{(-1)}$$

6 Important

$$A^{(-1)} = (A^*A)^{(-1)} A^*$$

Consequence If A^*A is invertible, then

$$A^{(-1)} = (A^*A)^{-1} \cdot A^*$$

and we can compute $A^{(-1)}$ without looking for a singular decomposition.

APPROXIMATION OF SOLUTIONS OF LIN. EQUATIONS

Let A be a matrix $k \times n$. Consider the system of equations

$$Ax = b \qquad x \in \mathbb{K}^n, \ b \in \mathbb{K}^k$$

If $\text{rank}(A|b) > \text{rank}(A)$ the system has no solution. In this case we want to find $x \in \mathbb{K}^n$ such that $\|Ax - b\|$ is minimal. ~~It means~~

Thm The function $\|Ax-b\|$ takes its minimum in $x = A^{(-1)} b$.

Example We have a system

$$x_1 + 2x_2 = 3$$
$$x_1 = 7$$
$$x_2 = 5$$

This system has no solution. Nevertheless, we are looking for x such that

$$(x_1 + 2x_2 - 3)^2 + (x_1 - 7)^2 + (x_2 - 5)^2$$

is minimal. The solution is

$$A^{(-1)} b = \begin{pmatrix} 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 28/6 \\ 1/3 \end{pmatrix}$$

Polar decomposition of matrices

Every square matrix $A \in Mat_{n \times n}(\mathbb{K})$ can be written in the form

$$A = R \cdot U$$

where R is selfadjoint and positive semidefinite and U is a ortonormal or unitary. If A is invertible, the decomposition is unique and

$$U = A\left(\sqrt{AA^*}\right)^{-1}$$

Proof: $A = PSQ^*$ is a singular decomposition. From here

$$A = \underbrace{PSP^*}_{\substack{R \\ \text{selfadjoint}}} \underbrace{PQ^*}_{\substack{U \\ \text{ortonormal or unitary}}}$$