$$\varphi : \underset{\mathcal{B}}{U} \longrightarrow \underset{\alpha}{V} \qquad A = (\varphi)_{\alpha\mathcal{B}}$$

$$\exists!?\ \varphi^* : V \longrightarrow U \quad \text{adj.}$$

$$(\varphi^*)_{\mathcal{B}\alpha} = \overline{A}^T \qquad f = (-)_\alpha \circ \varphi \circ (-)_{\mathcal{B}}^{-1}$$

$$\begin{tikzcd}
U \arrow[r, "\varphi"] \arrow[d, shift right, "(-)_{\mathcal{B}}"'] & V \arrow[d, "(-)_\alpha"] \\
\mathbb{C}^m \arrow[u, shift right, "(-)_{\mathcal{B}}^{-1}"'] \arrow[r, dashed] & \mathbb{C}^m
\end{tikzcd}$$

$$f(x) = Cx$$

$$f^*(y) = \overline{C}^T y$$

$$\varphi^* = (-)_{\mathcal{B}}^{-1} \circ f^* \circ (-)_\alpha$$

$$\langle \varphi(u), v \rangle_V = \langle (\varphi(u))_\alpha, (v)_\alpha \rangle_{\mathbb{C}^m}$$

$$= \langle (\underbrace{\varphi((x)^{-1}_\beta))_\alpha}_{f(x)}, (v)_\alpha \rangle_{\mathbb{C}^m} =$$

$$= \langle x, f^*((v)_\alpha) \rangle_{\mathbb{C}^n} =$$

$$= \langle \underbrace{(x)^{-1}_\beta}_{u}, (f^*((v)_\alpha))^{-1}_\beta \rangle_U =$$

$$= \langle u, \varphi^*(v) \rangle$$

$\varphi : U \to U, \quad V \subseteq U$

$\varphi(V) \subseteq V \Rightarrow \varphi(V^\perp) \subseteq V^\perp$

$x \in V^\perp \overset{?}{\Rightarrow} \langle \varphi(x), y \rangle = 0 \quad \forall y \in V$

$0 = \langle \underbrace{x}_{\in V^\perp}, \underbrace{\varphi(y)}_{\in V} \rangle$

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$\varphi : V \to V$, samosprzężony:

? 1. λ w.ł. $\Rightarrow \lambda \in \mathbb{R}$

$\varphi(v) = \lambda v$, $v \neq 0$

$\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle \varphi(v), v \rangle$

$= \langle v, \varphi(v) \rangle = \langle v, \lambda v \rangle = \langle v, v \rangle \bar{\lambda}$

$\lambda = \bar{\lambda} \in \mathbb{R}$

2. λ, μ vlastní čísla, $\lambda \neq \mu$,

$\underset{\neq 0}{\underset{\uparrow}{v}}$ $\underset{\neq 0}{\underset{\uparrow}{w}}$ $\overset{?}{\Longrightarrow} \langle v, w \rangle = 0$

$0 \neq \lambda$ $\lambda \cdot \langle v, w \rangle = \langle \lambda v, w \rangle =$

$= \langle \varphi(v), w \rangle = \langle v, \varphi(w) \rangle =$

$= \langle v, w \rangle \cdot \mu \Longrightarrow$

$(\underbrace{\lambda - \mu}_{\neq 0}) \cdot \langle v, w \rangle = 0 \Rightarrow v \perp w$

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indukcí dle $\dim U$

$\dim U = 1$ $\quad \varphi(u) = \lambda u$

$\alpha = (u)$ je báze U $\quad u \neq o \quad \|u\| = 1$

$$(\varphi)_\alpha = (\lambda)$$

Nechť to platí pro $\dim \leq n-1$.

$\varphi : U \to U$ $\exists$ vl. číslo λ_1 a ortn. vektor u_1 příslušný k λ_1; $\|u_1\| = 1$.

$\varphi(u_1) = \lambda_1 u_1$,

$V = [u_1] \oplus [u_1]^\perp$

$\varphi([u_1]^\perp)$ și invariant ??

$\subseteq [u_1]^\perp$

$\forall v \in [u_1]^\perp \Rightarrow \varphi(v) \in [u_1]^\perp$

$\langle \varphi(v), u_1 \rangle = \langle v, \underbrace{\varphi(u_1)}_{\lambda_1 u_1} \rangle =$

$= \underbrace{\langle v, u_1 \rangle}_{0} \cdot \lambda_1 = 0$

$\tilde{\varphi} = \varphi|_{[u_1]^\perp} : [u_1]^\perp \to [u_1]^\perp$

dle ind. předp. $\exists\ u_2, \dots, u_n$

$\beta = (u_2, \dots, u_n)$ ON BÁZE $[u_1]^\perp$

$\tilde{\varphi}(u_i) = \lambda_i u_i = \varphi(u_i)$

$\alpha = (u_1, \dots, u_n)$ je ON BÁZE U

$$(\varphi)_{\alpha,\alpha} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

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φ je samadj. $\Leftrightarrow$ $\varphi = \sum_{i=1}^{k} \lambda_i \cdot P_i$ v.č.

$\Leftarrow$: f, g sama. $\Rightarrow$ $f+g$ sam.

$\langle (f+g)(u), v\rangle = \langle f(u), v\rangle +$
$+ \langle g(u), v\rangle = \langle u, f(v)\rangle + \langle u, g(v)\rangle$
$= \langle u, (f+g)(u)\rangle$

φ je samoadj. $\Rightarrow \varphi = \sum_{i=1}^{k} \lambda_i P_i$

$\exists \underbrace{u_1, \dots}_{\lambda_1} \underbrace{\dots}_{\lambda_2} \underbrace{, u_n}_{\lambda_k}$ ON r.v.e.

$u_i^j, 1 \le i \le$ gm.nás. λ_j, $1 \le j \le k$

$\qquad\qquad\quad \| $
$\qquad\qquad$ alg.nás.

$S_j = [u_i^j, \dots]$ $\qquad U = S_1 \oplus \dots \oplus S_k$

$x \in U \qquad x = \sum_{j=1}^{k} \underbrace{x_j}_{\in S_j} \Rightarrow \varphi(x) = \sum \varphi(x_j) = \sum \lambda_j x_j$

($x_j = P_j$)

A je reálná sym. $n \times n$

$\exists P$ OG matice

$P^{-1} A P$ je diagonální ?

$\varphi(x) = Ax$, $\varphi : \mathbb{R}^n \to \mathbb{R}^n$

φ je samoadj.

$\exists \alpha$ ON báze

$$(\varphi)_{\alpha,\alpha} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$(\varphi)_{\alpha\alpha} = \underbrace{P_{\alpha \varepsilon}}_{P^{-1}} A_{\varepsilon,\varepsilon} \underbrace{P_{\varepsilon,\alpha}}_{P} \qquad P_{\varepsilon,\alpha} = \alpha$$

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$P^{-1}AP$ je diag., A matice
kv. formy

$$(y)_\varepsilon^T A (x)_\varepsilon = (y)_\alpha^T \, P_{\varepsilon\alpha}^T A P_{\varepsilon\alpha} \, (x)_\alpha$$

$$(x)_\varepsilon = P_{\varepsilon\alpha}(x)_\alpha \,, \quad (y)_\varepsilon = P_{\varepsilon\alpha}(y)_\alpha$$

$\varphi : U \to V, \quad \varphi^* : V \to U$

$\varphi^* \circ \varphi : U \to U$ same..

$\langle (\varphi^* \circ \varphi)(u), v \rangle = \langle \varphi(u), \varphi(v) \rangle$

$= \langle u, (\varphi^* \circ \varphi)(v) \rangle$

$\boxed{\varphi^{**} = \varphi}$

dub 23-18:01

$$\langle (\varphi^* \circ \varphi)(u), u \rangle =$$

$$= \langle \underline{\varphi(u)}, \underline{\varphi(u)} \rangle \geq 0$$

Nechť $0 \neq u \in U$, $(\varphi^* \circ \varphi)(u) = \lambda u$

$$\lambda \underbrace{\langle u, u \rangle}_{> 0} = \langle \lambda u, u \rangle = \langle \varphi^*(\varphi(u)), u \rangle =$$

$$= \langle \varphi(u), \varphi(u) \rangle \geq 0 \Rightarrow \underline{\underline{\lambda \geq 0}}$$

$$\operatorname{Ker}\varphi = \operatorname{Ker}(\varphi^* \circ \varphi)$$

$$x \in \operatorname{Ker}\varphi, \Rightarrow \varphi(x) = 0 \Rightarrow \underbrace{(\varphi^* \circ \varphi)(x)}_{=0}$$

$$\Rightarrow x \in \operatorname{Ker}(\varphi^* \circ \varphi).$$

$$y \in \operatorname{Ker}(\varphi^* \circ \varphi), \quad \varphi^*(\varphi(y)) = 0$$

$$0 = \langle 0, y \rangle = \langle \varphi^*(\varphi(y)), y \rangle = \langle \varphi(y), \varphi(y) \rangle$$

$$y \in \operatorname{Ker}\varphi \Leftarrow \varphi(y) = 0$$

A $k \times n$ $\varphi : \mathbb{K}^n \to \mathbb{K}^k$

$\varphi(x) = Ax$ $\varphi^* \circ \varphi : \mathbb{K}^n \to \mathbb{K}^n$

$\varphi^* \circ \varphi$ samoadj. nezáp. vl. čísla

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0, \lambda_{n+1} = \dots$

u_1 u_2 .. u_n ON BÁZE $= \lambda_n = 0$

u_j

$\varphi^* \circ \varphi(u_i) = \lambda_i u_i$ $\alpha = (u_1, \dots, u_n)$

$Q = (id)_{e,\alpha}$ $\underset{0}{\lambda_{n+1}} u_{n+1} = 0 \stackrel{?}{=} \varphi(u_{n+1})$

víme, že $(\varphi^* \circ \varphi)(u_j) = 0$ pro $n+1 \le j \le m$

$\operatorname{Ker} \varphi = \operatorname{Ker} \varphi^* \circ \varphi \implies \varphi(u_j) = 0$

pro $n+1 \le j \le m$. $\operatorname{Ker} \varphi = [u_{n+1}, \dots, u_m]$

$1 \le i \le n \implies \|\varphi(u_i)\|^2 = \langle \varphi(u_i), \varphi(u_i) \rangle =$

$= \langle (\varphi^* \circ \varphi)(u_i), u_i \rangle = \langle \lambda_i u_i, u_i \rangle =$

$= \lambda_i \langle u_i, u_i \rangle = \lambda_i > 0$ $\varphi(u_i) \neq 0$

$\|\varphi(u_i)\| = \sqrt{\lambda_i}$

$i \neq j,\ 1 \leq j \leq n \quad \langle \varphi(u_i), \varphi(u_j) \rangle =$

$= \langle (\varphi^* \circ \varphi)(u_i), u_j \rangle = \langle \lambda_i u_i, u_j \rangle$

$= \lambda_i \underbrace{\langle u_i, u_j \rangle}_{0} = 0$

$\varphi(u_i)$ jsou nenulové a navzájem kolmé.

(jsou lineárně nezávislé).

$(*)$

$v_i = \frac{\varphi(u_i)}{\|u_i\|} = \frac{\varphi(u_i)}{\sqrt{\lambda_i}} \Rightarrow \|v_i\| = 1$

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$v_1, \dots, v_n$ doplníme na ON bázi $\mathbb{K}^k$, označme ji β.

$P = (\mathrm{id})_{\varepsilon,\beta}$ | $\alpha = (u_1, \dots, u_n)$, $\beta = (v_1, \dots, v_k)$

$(\varphi)_{\varepsilon_k, \varepsilon_m} = A$

$(\varphi(u_n))_\beta = 0$

$$(\varphi)_{\beta,\alpha} = \left((\varphi(u_1))_\beta, \dots, (\varphi(u_n))_\beta \right)$$

$$\sqrt{\lambda_1}\, v_1 \qquad \sqrt{\lambda_n}\, v_n$$

$$S = \left(\begin{array}{ccc|c} \sqrt{\lambda_1} & & & \\ & \ddots & & 0 \\ & & \sqrt{\lambda_n} & \\ \hline & 0 & & \end{array} \right)$$

$$A = (\varphi)_{\varepsilon_1, \varepsilon_n} = (id)_{\varepsilon, \beta} (\varphi)_{\beta, \alpha} (id)_{\alpha, \varepsilon_n}$$

$$= P \cdot S \cdot Q^*$$

P, S, $Q^* = Q^{-1}$

$$(A^{(-1)})^{(-1)} = A \qquad S = \left(\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$A = PSQ^*$$

$$A^{(-1)} = Q \left(\begin{array}{c|c} D^{-1} & 0 \\ \hline 0 & 0 \end{array}\right) P^*$$

$$(A^{(-1)})^{(-1)} = P \left(\begin{array}{c|c} (D^{-1})^{-1} & 0 \\ \hline 0 & 0 \end{array}\right) Q^*$$

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$$③ \ (A^{(-1)}A)^* = \left(Q \left(\begin{array}{c|c} D^{-1} & 0 \\ \hline 0 & 0 \end{array} \right) P^* \cdot P \cdot \left(\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array} \right) \cdot Q^* \right)^* = \left(Q \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right) Q^* \right)^*$$

$$= Q^{**} \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right)^* Q^* = Q \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right) Q^*$$

$$= \underline{A^{(-1)} \cdot A}.$$

$$A A^{(-1)} A = P S Q^* Q (S^{-1}) P^*$$

$$\cdot P \cdot S \cdot Q^* = P \begin{pmatrix} \boxed{D} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \cdot Q^* = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} Q^* =$$

$$= P \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} Q^* = \underline{A}$$

$\varphi : \mathbb{K}^n \to \mathbb{K}^k, \ \varphi(x) = Ax$

$\varphi^{(-1)} : \mathbb{K}^k \to \mathbb{K}^n, \ \varphi^{(-1)}(y) = A^{(-1)}y$

$\varphi \circ \varphi^{(-1)} : \mathbb{K}^k \to \mathbb{K}^k$ je ktomná

projekce do $\mathrm{Im}\,\varphi$

$\mathrm{Im}(\varphi \circ \varphi^{(-1)}) \overset{\supseteq}{\subseteq} \mathrm{Im}\,\varphi$

$x \in \mathbb{K}^n, \ Ax \in \mathrm{Im}\,\varphi$

$\underline{A}x = \underline{A A^{(-1)} A}x$

$\varphi \circ \varphi^{(-1)}/_{\mathrm{Im}\,\varphi} =$

$= id_{\mathrm{Im}\,\varphi}$

$\varphi \circ \varphi^{(-1)} : \mathbb{K}^{k} \to \mathbb{K}^{k}$ Kolmá projekce?

$y \in K^{k}$ $\quad y - (\varphi \circ \varphi^{(-1)})(y) \in \operatorname{Im} \varphi^{\perp}$

$x \in K^{n}, \; Ax \in \operatorname{Im} \varphi$

$(Ax)^{T} \cdot (y - AA^{(-1)} y) = x^{T} A^{T} y - x^{T} A^{T} (A A^{(-1)}) y = 0$

$A^{T} (A A^{(-1)})^{T} = (\underbrace{A A^{(-1)} A}_{A^{T}})^{T}$

$$(A^*A)^{(-1)}A^* = (Q S^* P^* P S Q^*)^{(-1)}$$

$$\cdot A^* = \left(Q \left(\begin{array}{ccc|c} \sigma_1^2 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_n^2 & \\ \hline & 0 & & 0 \end{array}\right) Q^* \right)^{(-1)} \cdot A^*$$

$$= \left(Q \cdot \left(\begin{array}{ccc|c} \frac{1}{\sigma_1^2} & & & \\ & \ddots & & 0 \\ & & \frac{1}{\sigma_n^2} & \\ \hline & 0 & & 0 \end{array}\right) Q^* \cdot Q S^* P^* \right.$$

$$S^* = \left(\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_n & \\ \hline & 0 & & 0 \end{array}\right)$$

$$= Q \left(\begin{array}{ccc|c} \frac{1}{\sigma_1} & & & \\ & \ddots & & 0 \\ & & \frac{1}{\sigma_n} & \\ \hline & 0 & & 0 \end{array}\right) \cdot P^* = A^{(-1)}$$