M6201 Příklady - lineární systémy

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8. února 2020

Příklad

Najděte obecné řešení lineárního systému

$$\begin{aligned}\dot{x}_1 &= 7x_1 + 3x_2\\ \dot{x}_2 &= x_1 + 5x_2.\end{aligned}$$

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Řešení:

$A = \begin{pmatrix} 7 & 3 \\ 1 & 5 \end{pmatrix}$

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$\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & 3 \\ 1 & 5-\lambda \end{vmatrix} = \lambda^2 - 12\lambda + 32 = (\lambda - 8)(\lambda - 4) = 0$

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$\lambda_1 = 8: \quad \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} \quad \boldsymbol{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\lambda_2 = 4: \quad \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad T = (\boldsymbol{v}_1, \boldsymbol{v}_2) = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}.$

$$
\begin{aligned}
\boldsymbol{x} &= \boldsymbol{T}\boldsymbol{u} \quad \Rightarrow \quad \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \quad \Rightarrow \quad \boldsymbol{T}\dot{\boldsymbol{u}} = \boldsymbol{A}\boldsymbol{T}\boldsymbol{u} \\
\dot{\boldsymbol{u}} &= \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}\boldsymbol{u} = \tfrac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 7 & 3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\boldsymbol{u}
\end{aligned}
$$

$$
\begin{aligned}
\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} &= \begin{pmatrix} 8 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
\dot{u}_1 &= 8u_1 \\
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Řešení: $u_1(t) = c_1 e^{8t}$, $u_2(t) = c_2 e^{4t}$

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$\begin{pmatrix} 3c_1 e^{8t} + c_2 e^{4t} \\ c_1 e^{8t} - c_2 e^{4t} \end{pmatrix}.$

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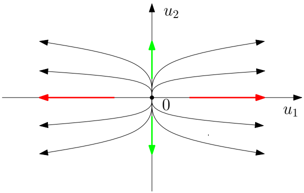

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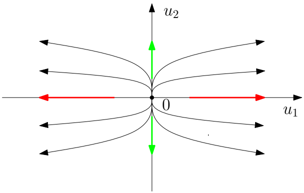

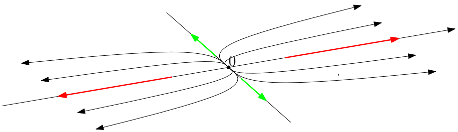

Příklad

Najděte obecné řešení lineárního systému

$$\begin{aligned} \dot{x}_1 &= 4x_1 + 5x_2 \\ \dot{x}_2 &= -x_1 - 2x_2. \end{aligned}$$

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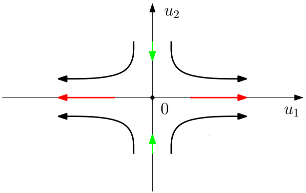

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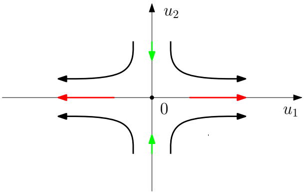

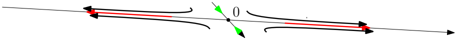

Příklad

Najděte obecné řešení lineárního systému

$$\begin{aligned} \dot{x}_1 &= -x_1 + 6x_2 \\ \dot{x}_2 &= 2x_1 + 3x_2. \end{aligned}$$

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$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 6 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 15 = (\lambda - 5)(\lambda + 3) = 0$$

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$\lambda_1 = 5: \quad \begin{pmatrix} -6 & 6 \\ 2 & -2 \end{pmatrix} \quad \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -3: \quad \begin{pmatrix} 2 & 6 \\ 2 & 6 \end{pmatrix} \quad \boldsymbol{v}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

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$\boldsymbol{x} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 e^{5t} \\ c_2 e^{-3t} \end{pmatrix} = \begin{pmatrix} c_1 e^{5t} + 3c_2 e^{-3t} \\ c_1 e^{5t} - c_2 e^{-3t} \end{pmatrix}.$

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$A = \begin{pmatrix}3 & -2\\ 2 & -1\end{pmatrix}$

$\det(A-\lambda I) = \begin{vmatrix}3-\lambda & -2\\ 2 & -1-\lambda\end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2 = 0$

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$\lambda_1 = 1: \quad \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \quad \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

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$A\boldsymbol{v}_2 = \lambda_1 \boldsymbol{v}_2 + \boldsymbol{v}_1: \quad \begin{pmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \end{pmatrix} \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$

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\dot{u}_1 &= u_1 + u_2 \\
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\end{aligned}$$

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Řešení: $u_2(t) = c_2 e^t$, dosazením do první rovnice máme

$$\begin{aligned} \dot{u}_1 - u_1 &= c_2 e^t \\ \dot{u}_1 e^{-t} - u_1 e^{-t} &= \tfrac{\mathrm{d}u_1 e^{-t}}{\mathrm{d}t} = c_2. \end{aligned}$$

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Odtud $u_1 = (c_1 + c_2 t)e^t$

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Odtud $u_1 = (c_1 + c_2 t)e^t$

$\boldsymbol{x} = \boldsymbol{T} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (\boldsymbol{v}_1, \boldsymbol{v}_2) \begin{pmatrix} (c_1 + c_2 t)e^t \\ c_2 e^t \end{pmatrix} =$

$(c_1 + c_2 t)e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} (c_1 + c_2 t + c_2)e^t \\ (c_1 + c_2 t + \frac{1}{2} c_2)e^t \end{pmatrix}.$

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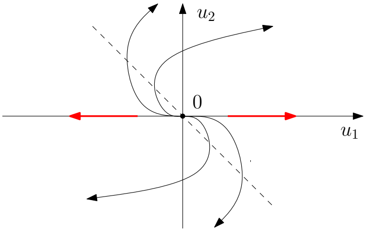

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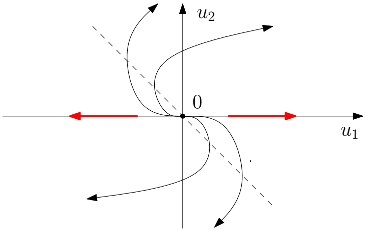

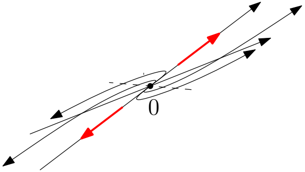

Příklad

Najděte obecné řešení lineárního systému

$$\begin{aligned} \dot{x}_1 &= x_1 - 4x_2 \\ \dot{x}_2 &= x_1 + x_2. \end{aligned}$$

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$$\boldsymbol{x} = \begin{pmatrix} 2i & -2i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{(1+2i)t} \\ c_2 e^{(1-2i)t} \end{pmatrix}.$$

Označme $\boldsymbol{u} = \operatorname{Re} \boldsymbol{v}_1$ a $\boldsymbol{w} = \operatorname{Im} \boldsymbol{v}_1$, tedy $\boldsymbol{v}_1 = \boldsymbol{u} + i\boldsymbol{w}$ a $\boldsymbol{v}_2 = \boldsymbol{u} - i\boldsymbol{w}$.

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Pak platí

$$
\begin{aligned}
\boldsymbol{x} &= (\boldsymbol{u} + i\boldsymbol{w}, \boldsymbol{u} - i\boldsymbol{w}) \begin{pmatrix} c_1 e^t(\cos 2t + i \sin 2t) \\ c_2 e^t(\cos 2t - i \sin 2t) \end{pmatrix} \\
&= c_1 e^t \cos 2t\boldsymbol{u} - c_1 e^t \sin 2t\boldsymbol{w} + c_2 e^t \cos 2t\boldsymbol{u} - c_2 e^t \sin 2t\boldsymbol{w} \\
&\quad + ic_1 e^t \cos 2t\boldsymbol{w} + ic_1 e^t \sin 2t\boldsymbol{u} - ic_2 e^t \cos 2t\boldsymbol{w} - ic_2 e^t \sin 2t\boldsymbol{u} \\
&= (c_1 + c_2)e^t \cos 2t\boldsymbol{u} - (c_1 + c_2)e^t \sin 2t\boldsymbol{w} \\
&\quad + i(c_1 - c_2)e^t \cos 2t\boldsymbol{w} + i(c_1 - c_2)e^t \sin 2t\boldsymbol{u} \\
&= e^t(k_1 \cos 2t + k_2 \sin 2t)\boldsymbol{u} + e^t(k_2 \cos 2t - k_1 \sin 2t)\boldsymbol{w},
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kde $k_1 = (c_1 + c_2)$ a $k_2 = i(c_1 - c_2)$.

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kde $k_1 = (c_1 + c_2)$ a $k_2 = i(c_1 - c_2)$.
Maticově lze řešení zapsat jako $\boldsymbol{x} = \boldsymbol{T}\boldsymbol{R} \begin{pmatrix} k_1 e^t \\ k_2 e^t \end{pmatrix}$,
kde $\boldsymbol{T} = (\boldsymbol{u}, \boldsymbol{w})$ a $\boldsymbol{R} = \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix}$ je matice rotace o úhel $-2t$ (po směru hodinových ručiček).

$$\boldsymbol{x} = \boldsymbol{T}\boldsymbol{R}\begin{pmatrix} k_1 e^t \\ k_2 e^t \end{pmatrix},$$

kde $\boldsymbol{T} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ a $\boldsymbol{R} = \begin{pmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{pmatrix}$

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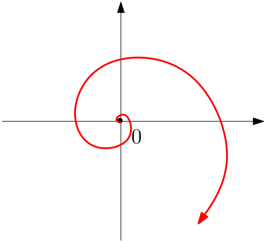

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