

Lecture 12: Strong coupling

Strong coupling

$$\gamma_1 = \gamma_2, \quad \omega_{0,1} \approx \omega_{0,2}, \quad \Omega_1 \approx \Omega_2$$

Secular approximation not applicable:

$$\mathcal{H} = +\omega_{0,1} \mathcal{I}_{1z} + \omega_{0,2} \mathcal{I}_{2z} + \pi J (2\mathcal{I}_{1z}\mathcal{I}_{2z} + 2\mathcal{I}_{1x}\mathcal{I}_{2x} + 2\mathcal{I}_{1y}\mathcal{I}_{2y})$$

\mathcal{I}_{1z} and \mathcal{I}_{2z} do not commute with $2\mathcal{I}_{1x}\mathcal{I}_{2x}$ and $2\mathcal{I}_{1y}\mathcal{I}_{2y}$:

$$[\mathcal{I}_{1z}, 2\mathcal{I}_{1x}\mathcal{I}_{2x}] = 2[\mathcal{I}_{1z}, \mathcal{I}_{1x}]\mathcal{I}_{2x} = i2\mathcal{I}_{1y}\mathcal{I}_{2x}$$

$$[\mathcal{I}_{1z}, 2\mathcal{I}_{1y}\mathcal{I}_{2y}] = 2[\mathcal{I}_{1z}, \mathcal{I}_{1y}]\mathcal{I}_{2y} = -i2\mathcal{I}_{1x}\mathcal{I}_{2y}$$

$$[\mathcal{I}_{2z}, 2\mathcal{I}_{1x}\mathcal{I}_{2x}] = 2\mathcal{I}_{1x}[\mathcal{I}_{2z}, \mathcal{I}_{2x}] = i2\mathcal{I}_{1x}\mathcal{I}_{2y}$$

$$[\mathcal{I}_{2z}, 2\mathcal{I}_{1y}\mathcal{I}_{2y}] = 2\mathcal{I}_{1y}[\mathcal{I}_{2z}, \mathcal{I}_{2y}] = -i2\mathcal{I}_{1y}\mathcal{I}_{2x}$$

Effects of $2\mathcal{I}_{1x}\mathcal{I}_{2x}$, $2\mathcal{I}_{1y}\mathcal{I}_{2y}$ and \mathcal{I}_{1z} , \mathcal{I}_{2z}

cannot be analyzed separately in any order

Strong coupling

$$\mathcal{H} = +\omega_{0,1}\mathcal{I}_{1z} + \omega_{0,2}\mathcal{I}_{2z} + \pi J (2\mathcal{I}_{1z}\mathcal{I}_{2z} + 2\mathcal{I}_{1x}\mathcal{I}_{2x} + 2\mathcal{I}_{1y}\mathcal{I}_{2y})$$

Hamiltonian not diagonal:

$$\mathcal{H} = \frac{\pi}{2} \begin{pmatrix} \Sigma + J & 0 & 0 & 0 \\ 0 & \Delta - J & 2J & 0 \\ 0 & 2J & -\Delta - J & 0 \\ 0 & 0 & 0 & -\Sigma + J \end{pmatrix}$$

$$\Sigma = (\omega_{0,1} + \omega_{0,2})/\pi$$

$$\Delta = (\omega_{0,1} - \omega_{0,2})/\pi$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

NOT stationary states (eigenfunctions of \mathcal{H})

New basis \Rightarrow diagonalized Hamiltonian

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{\frac{1}{2} + \frac{\Delta}{2\sqrt{\Delta^2 + 4J^2}}} \\ \sqrt{\frac{1}{2} - \frac{\Delta}{2\sqrt{\Delta^2 + 4J^2}}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\sqrt{\frac{1}{2} - \frac{\Delta}{2\sqrt{\Delta^2 + 4J^2}}} \\ \sqrt{\frac{1}{2} + \frac{\Delta}{2\sqrt{\Delta^2 + 4J^2}}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Delta = (\omega_{0,1} - \omega_{0,2})/\pi$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ c_\xi \\ s_\xi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -s_\xi \\ c_\xi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

stationary states (eigenfunctions of \mathcal{H}')

Diagonalized Hamiltonian

$$\mathcal{H}' = \frac{\pi}{2} \begin{pmatrix} \Sigma + J & 0 & 0 & 0 \\ 0 & \sqrt{\Delta^2 + 4J^2} - J & 0 & 0 \\ 0 & 0 & -\sqrt{\Delta^2 + 4J^2} - J & 0 \\ 0 & 0 & 0 & -\Sigma + J \end{pmatrix}$$

$$= \frac{\omega'_{0,1}}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \frac{\omega'_{0,1}}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \pi J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{H}' = \omega'_{0,1} \mathcal{I}_{1z} + \omega'_{0,2} \mathcal{I}_{2z} + \pi J \cdot 2 \mathcal{I}_{1z} \mathcal{I}_{2z}$$

$$\omega'_{0,1} = \frac{1}{2} \left(\omega_{0,1} + \omega_{0,2} + \sqrt{(\omega_{0,1} - \omega_{0,2})^2 + 4\pi^2 J^2} \right)$$

$$\omega'_{0,2} = \frac{1}{2} \left(\omega_{0,1} + \omega_{0,2} - \sqrt{(\omega_{0,1} - \omega_{0,2})^2 + 4\pi^2 J^2} \right)$$

$\hat{\rho}$ and \hat{M}_+ in the new basis

$$\mathcal{I}'_{1y} + \mathcal{I}'_{2y} = c_\xi(\mathcal{I}_{1y} + \mathcal{I}_{2y}) + s_\xi(2\mathcal{I}_{1z}\mathcal{I}_{2y} - 2\mathcal{I}_{1y}\mathcal{I}_{2z})$$

$$\mathcal{I}'_{1x} + \mathcal{I}'_{2x} = c_\xi(\mathcal{I}_{1x} + \mathcal{I}_{2x}) + s_\xi(2\mathcal{I}_{1z}\mathcal{I}_{2x} - 2\mathcal{I}_{1x}\mathcal{I}_{2z})$$

$$\begin{aligned} \mathcal{I}'_{1+} + \mathcal{I}'_{2+} &= c_\xi(\mathcal{I}_{1x} + \mathcal{I}_{2x} + i\mathcal{I}_{1y} + i\mathcal{I}_{2y}) \\ &+ s_\xi(2\mathcal{I}_{1z}\mathcal{I}_{2x} - 2\mathcal{I}_{1x}\mathcal{I}_{2z} + i2\mathcal{I}_{1z}\mathcal{I}_{2y} - i2\mathcal{I}_{1y}\mathcal{I}_{2z}) \end{aligned}$$

Contribution	$\frac{2}{\kappa}\hat{\rho}'_1(\text{b})$	$\omega'_{0,1}\mathcal{I}_{1z}$ →	$\pi J \cdot 2\mathcal{I}_{1z}\mathcal{I}_{2z}$ →
\mathcal{I}_{1y}	$+c_\xi$	$+c_\xi c'_1$	$+c_\xi c'_1 c_J - s_\xi s'_1 s_J$
$2\mathcal{I}_{1y}\mathcal{I}_{2z}$	$+s_\xi$	$+s_\xi c'_1$	$+s_\xi c'_1 c_J - c_\xi s'_1 s_J$
\mathcal{I}_{1x}	0	$-c_\xi s'_1$	$-c_\xi s'_1 c_J - s_\xi c'_1 s_J$
$2\mathcal{I}_{1y}\mathcal{I}_{2z}$	0	$-s_\xi s'_1$	$-s_\xi s'_1 c_J - c_\xi c'_1 s_J$

Signal of a strongly coupled pair

Contribution	$\frac{2}{\kappa}\tilde{\rho}'_1(\text{b})$	$\omega'_{0,1}\mathcal{I}_{1z}$ →	$\pi J \cdot 2\mathcal{I}_{1z}\mathcal{I}_{2z}$ →
\mathcal{I}_{1y}	$+c_\xi$	$+c_\xi c'_1$	$+c_\xi c'_1 c_J - s_\xi s'_1 s_J$
$2\mathcal{I}_{1y}\mathcal{I}_{2z}$	$+s_\xi$	$+s_\xi c'_1$	$+s_\xi c'_1 c_J - c_\xi s'_1 s_J$
\mathcal{I}_{1x}	0	$-c_\xi s'_1$	$-c_\xi s'_1 c_J - s_\xi c'_1 s_J$
$2\mathcal{I}_{1y}\mathcal{I}_{2z}$	0	$-s_\xi s'_1$	$-s_\xi s'_1 c_J - c_\xi c'_1 s_J$

Contrib.	$\text{Tr}\left\{\frac{2}{\kappa}\tilde{\rho}'_1(t)\mathcal{I}'_{1+}\right\}$
\mathcal{I}_{1y}	$\left. \begin{array}{l} +ic_\xi^2 c'_1 c_J - ic_\xi s_\xi s'_1 s_J \\ +is_\xi^2 c'_1 c_J - ic_\xi s_\xi s'_1 s_J \end{array} \right\} = i \left(c'_1 c_J - \frac{2J}{\sqrt{4J^2 + \Delta^2}} s'_1 s_J \right)$
$2\mathcal{I}_{1y}\mathcal{I}_{2z}$	
\mathcal{I}_{1x}	$\left. \begin{array}{l} -c_\xi^2 s'_1 c_J - c_\xi s_\xi c'_1 s_J \\ -s_\xi^2 s'_1 c_J - c_\xi s_\xi c'_1 s_J \end{array} \right\} = - \left(s'_1 c_J + \frac{2J}{\sqrt{4J^2 + \Delta^2}} c'_1 s_J \right)$
$2\mathcal{I}_{1y}\mathcal{I}_{2z}$	

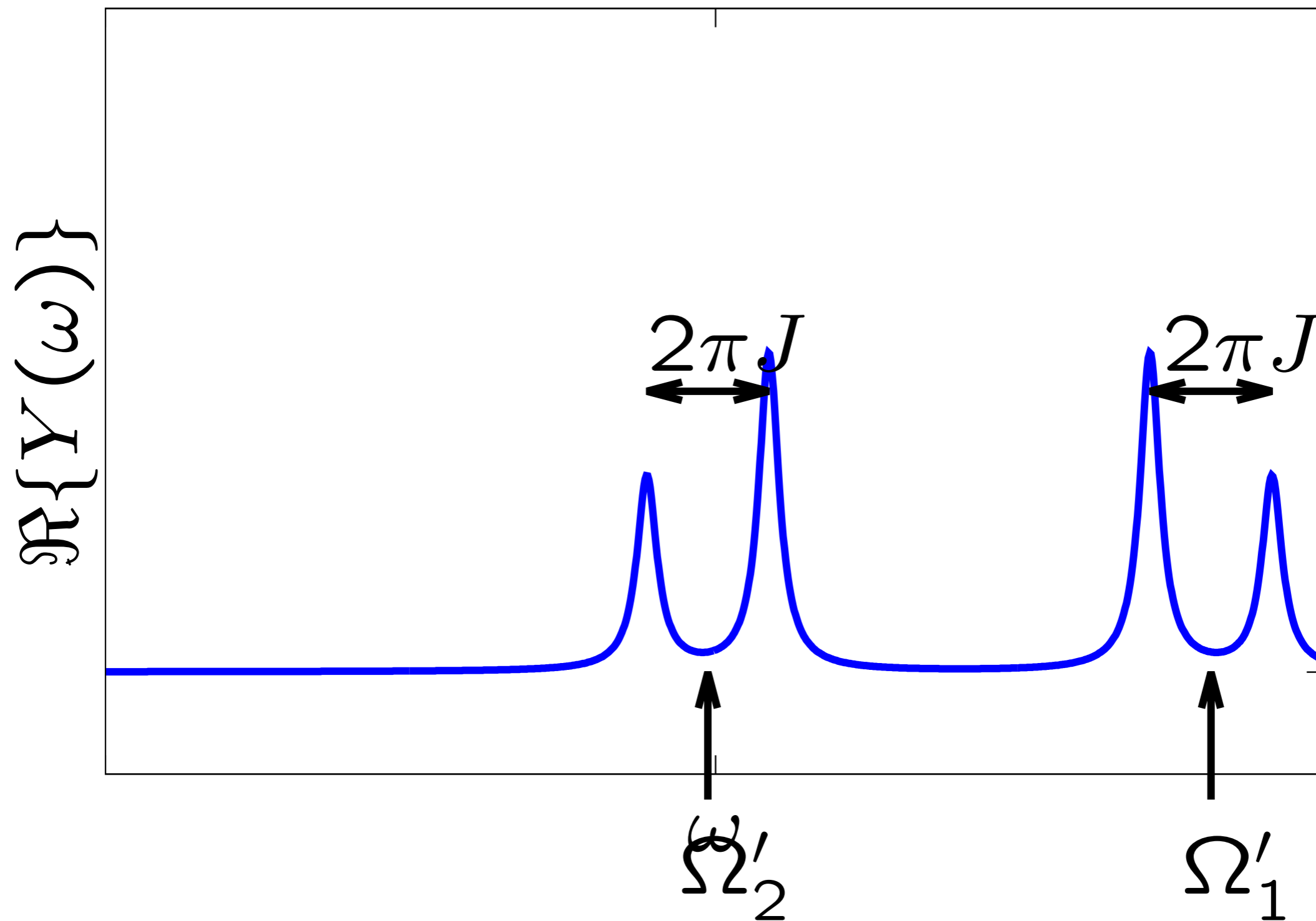
Signal of a strongly coupled pair

$$\begin{aligned}\Re\{Y(\omega)\} &= \left(1 - \frac{J}{\sqrt{\Delta^2 + 4J^2}}\right) \frac{\mathcal{N}\gamma^2\hbar^2_{B0}}{16k_{\text{B}}T} \frac{\bar{R}_2}{\bar{R}_2^2 + (\omega - \Omega'_1 - \pi J)^2} \\ &+ \left(1 + \frac{J}{\sqrt{\Delta^2 + 4J^2}}\right) \frac{\mathcal{N}\gamma^2\hbar^2_{B0}}{16k_{\text{B}}T} \frac{\bar{R}_2}{\bar{R}_2^2 + (\omega - \Omega'_1 - \pi J)^2} \\ &+ \left(1 + \frac{J}{\sqrt{\Delta^2 + 4J^2}}\right) \frac{\mathcal{N}\gamma^2\hbar^2_{B0}}{16k_{\text{B}}T} \frac{\bar{R}_2}{\bar{R}_2^2 + (\omega - \Omega'_2 + \pi J)^2} \\ &+ \left(1 - \frac{J}{\sqrt{\Delta^2 + 4J^2}}\right) \frac{\mathcal{N}\gamma^2\hbar^2_{B0}}{16k_{\text{B}}T} \frac{\bar{R}_2}{\bar{R}_2^2 + (\omega - \Omega'_2 + \pi J)^2}\end{aligned}$$

$$\Omega'_1 = \frac{1}{2} \left(\Omega_1 + \Omega_2 + \sqrt{(\Omega_1 - \Omega_2)^2 + 4\pi^2 J^2} \right)$$

$$\Omega'_2 = \frac{1}{2} \left(\Omega_1 + \Omega_2 - \sqrt{(\Omega_1 - \Omega_2)^2 + 4\pi^2 J^2} \right)$$

Spectrum of a strongly coupled pair

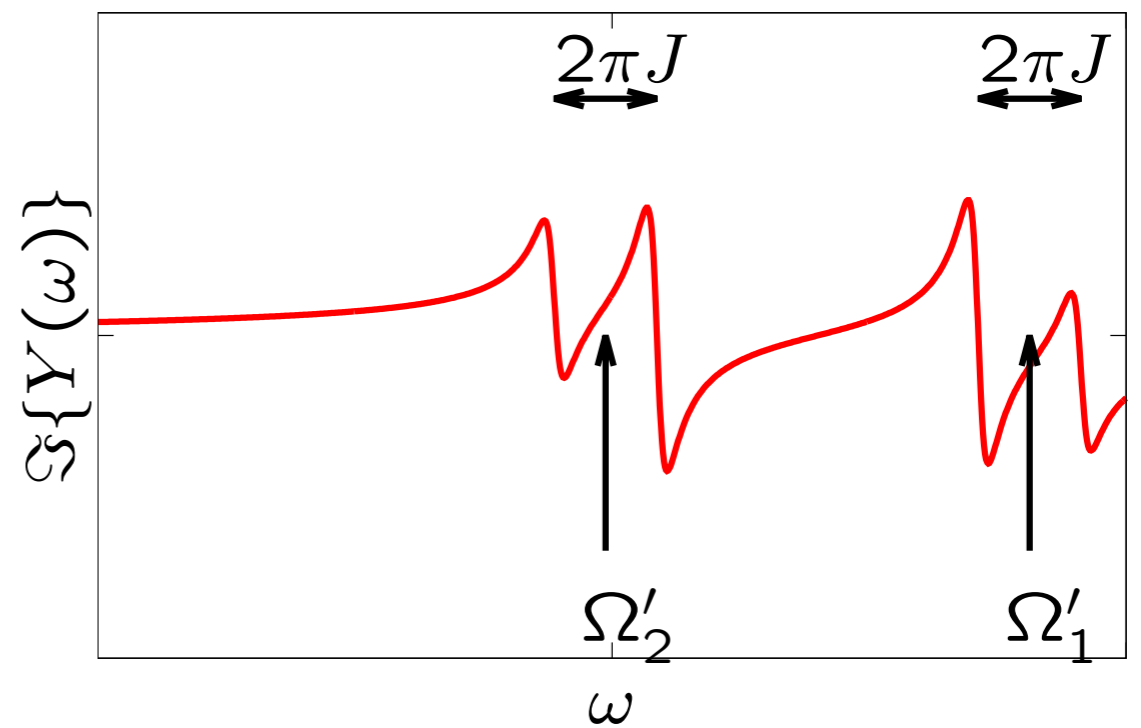
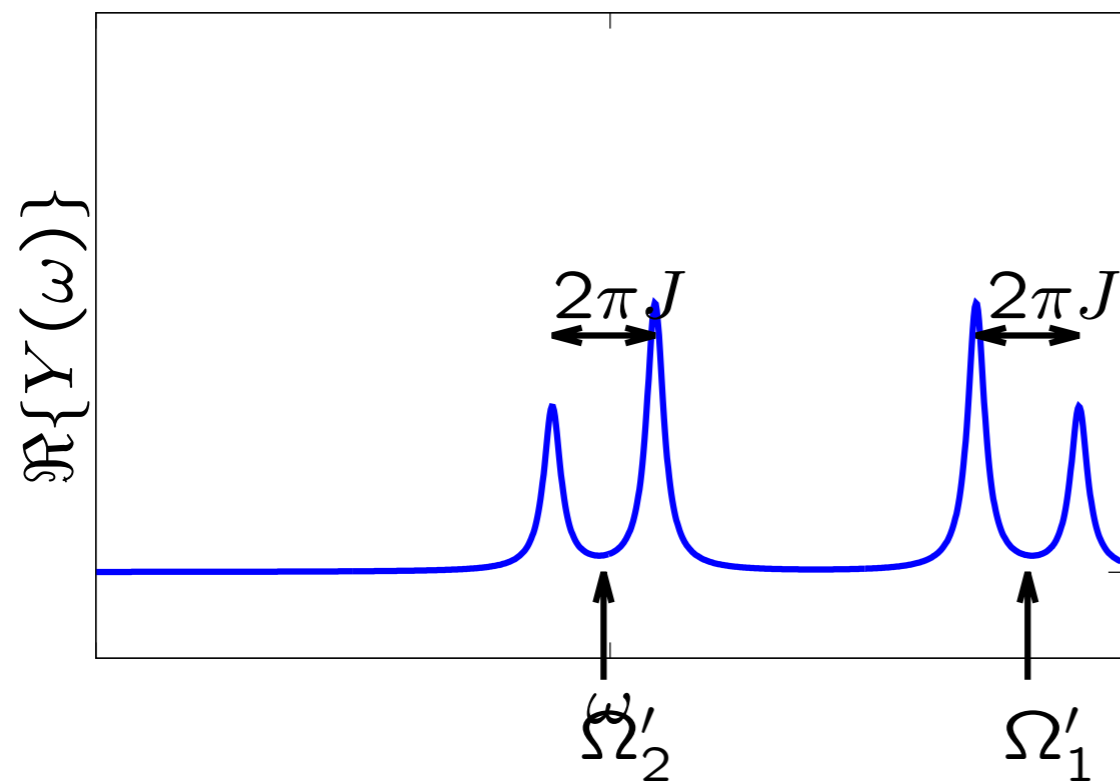


Strong vs. weak J -coupling

- Centers of doublets shifted from Ω_1 and Ω_2 by $\pm \left(\Omega_1 - \Omega_2 - \sqrt{(\Omega_1 - \Omega_2)^2 + 4\pi^2 J^2} \right) / 2$
- Intensities of inner/outer peaks increased/decreased by $2\pi J / \sqrt{(\Omega_1 - \Omega_2)^2 + 4\pi^2 J^2}$
- $\sqrt{(\Omega_1 - \Omega_2)^2 + 4\pi^2 J^2}$ makes the difference
 $|\Omega_1 - \Omega_2| \gg 2\pi|J|$ weak

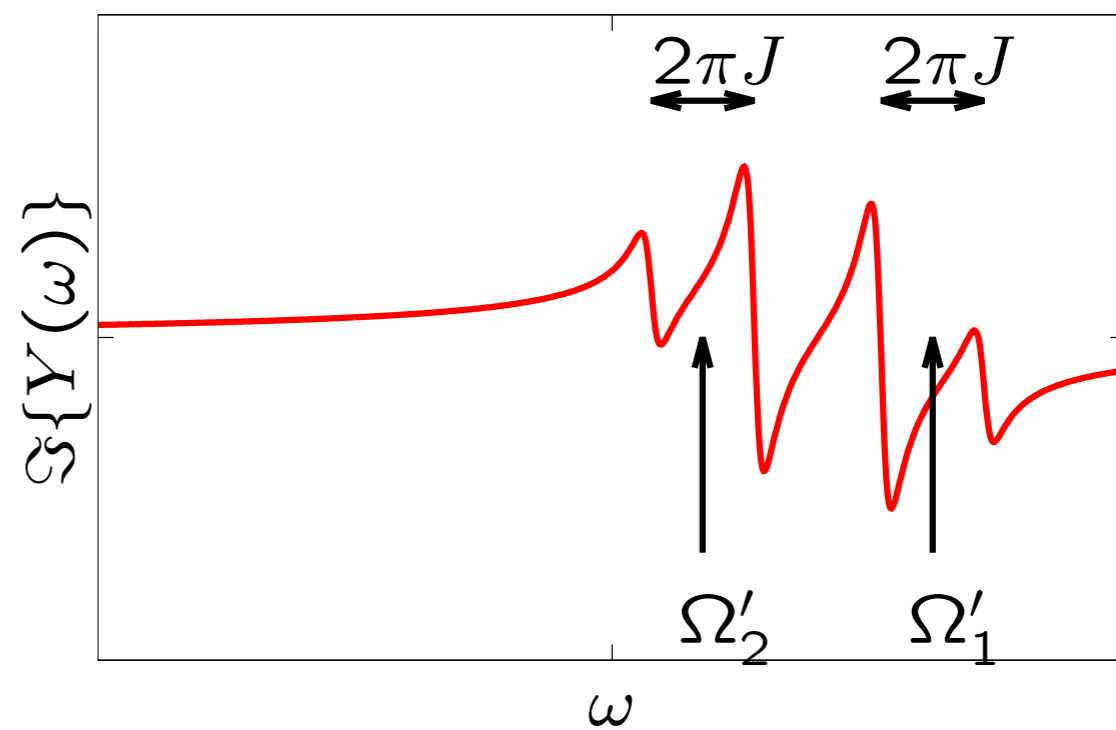
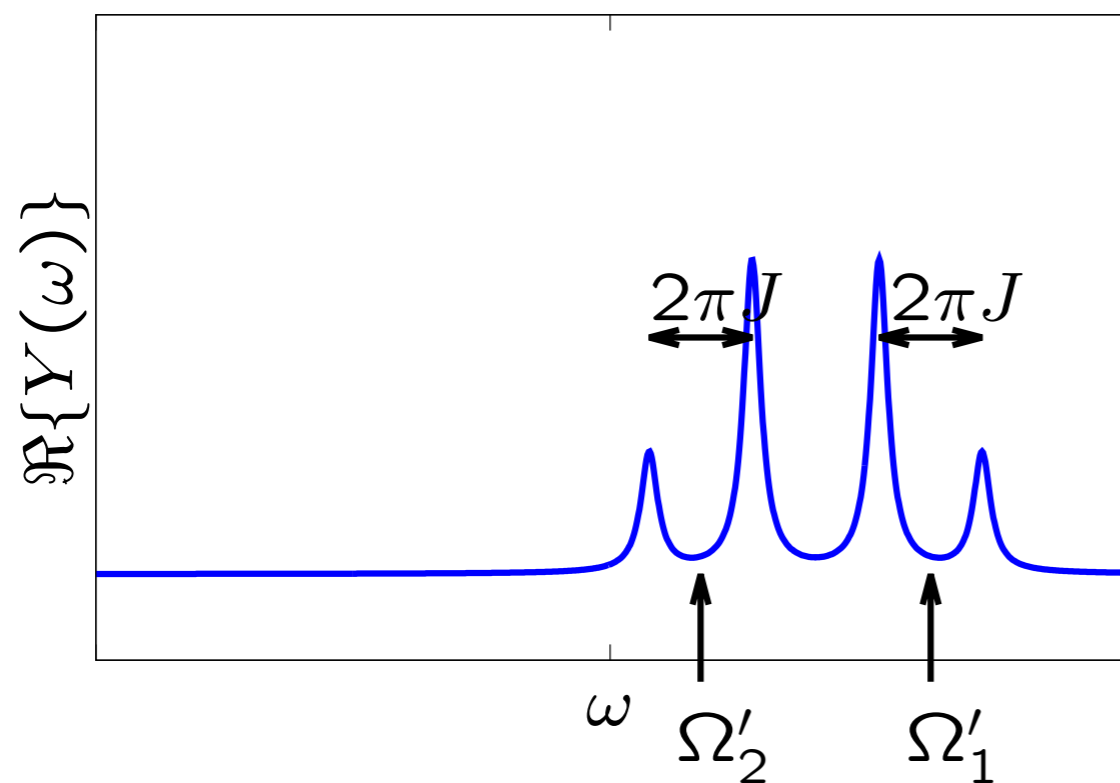
Spectrum of a strongly coupled pair

$$\Omega_1 - \Omega_2 = 8.0\pi J$$



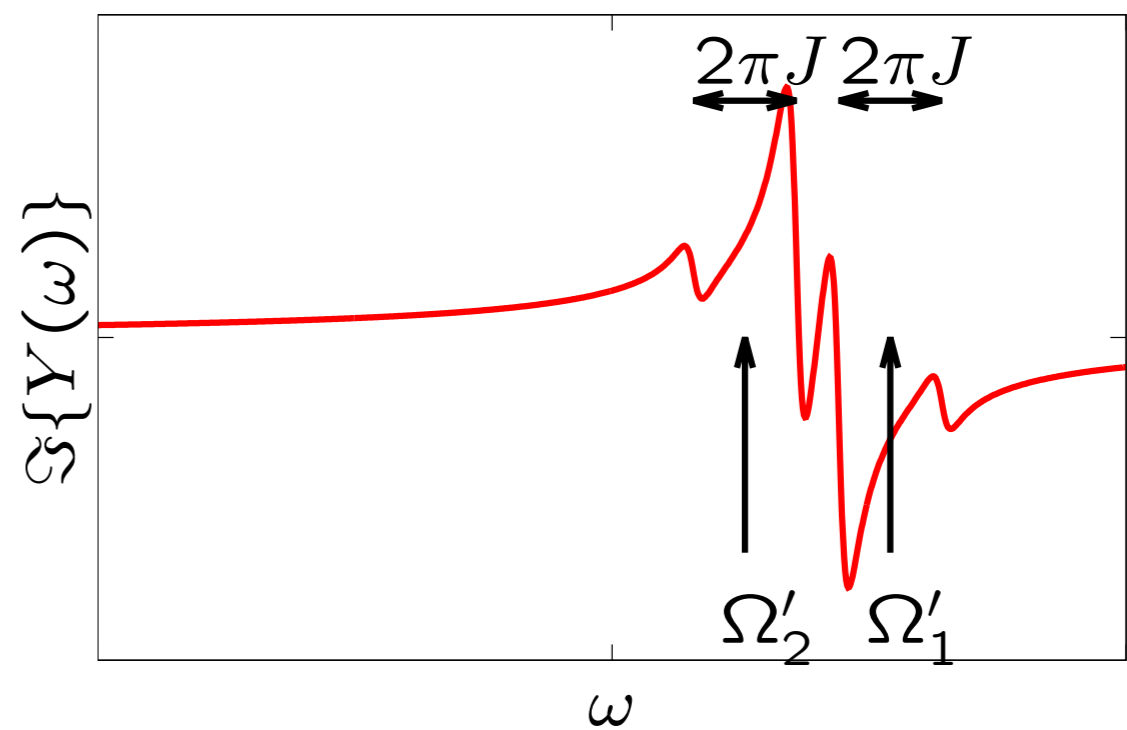
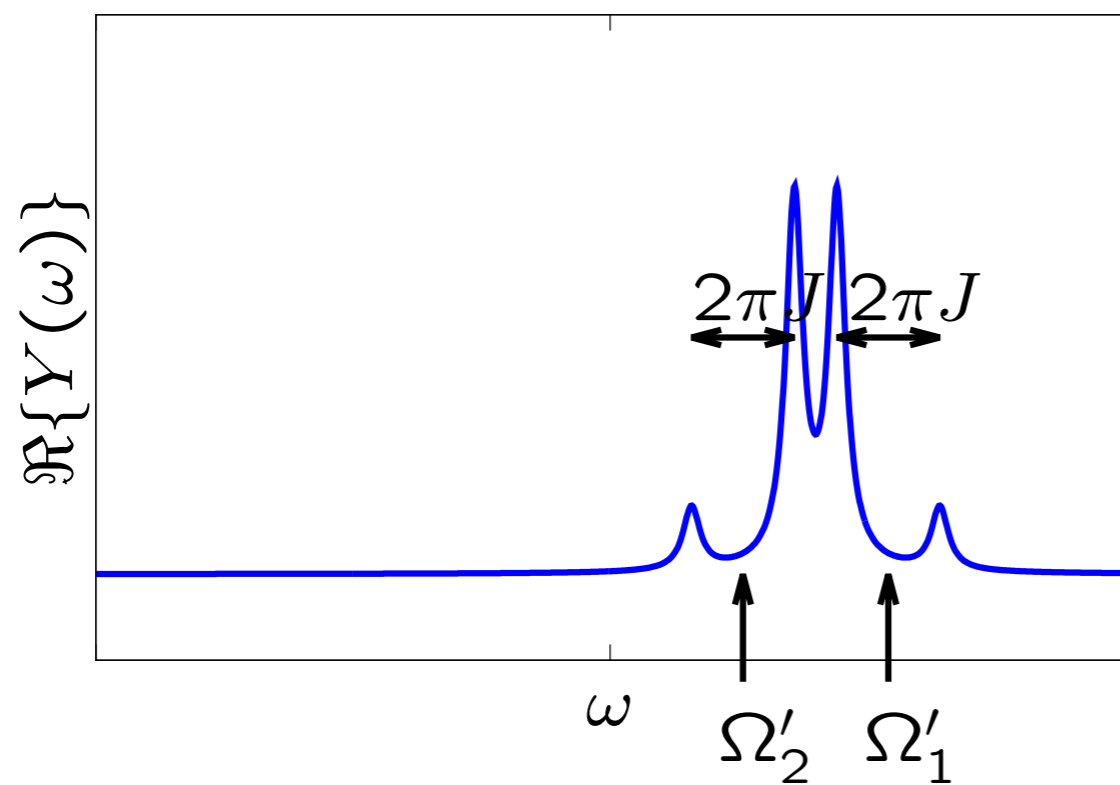
Spectrum of a strongly coupled pair

$$\Omega_1 - \Omega_2 = 4.0\pi J$$



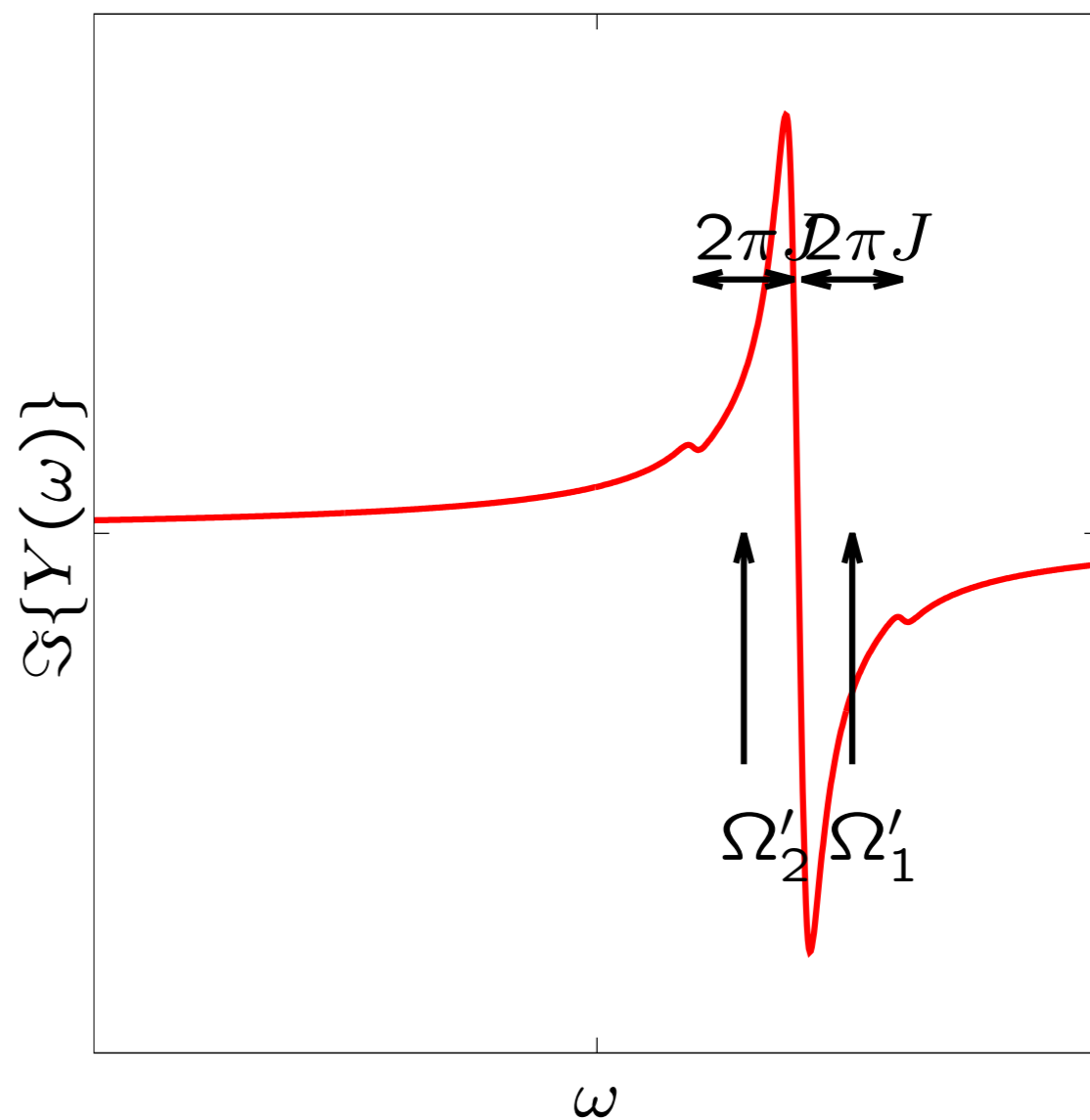
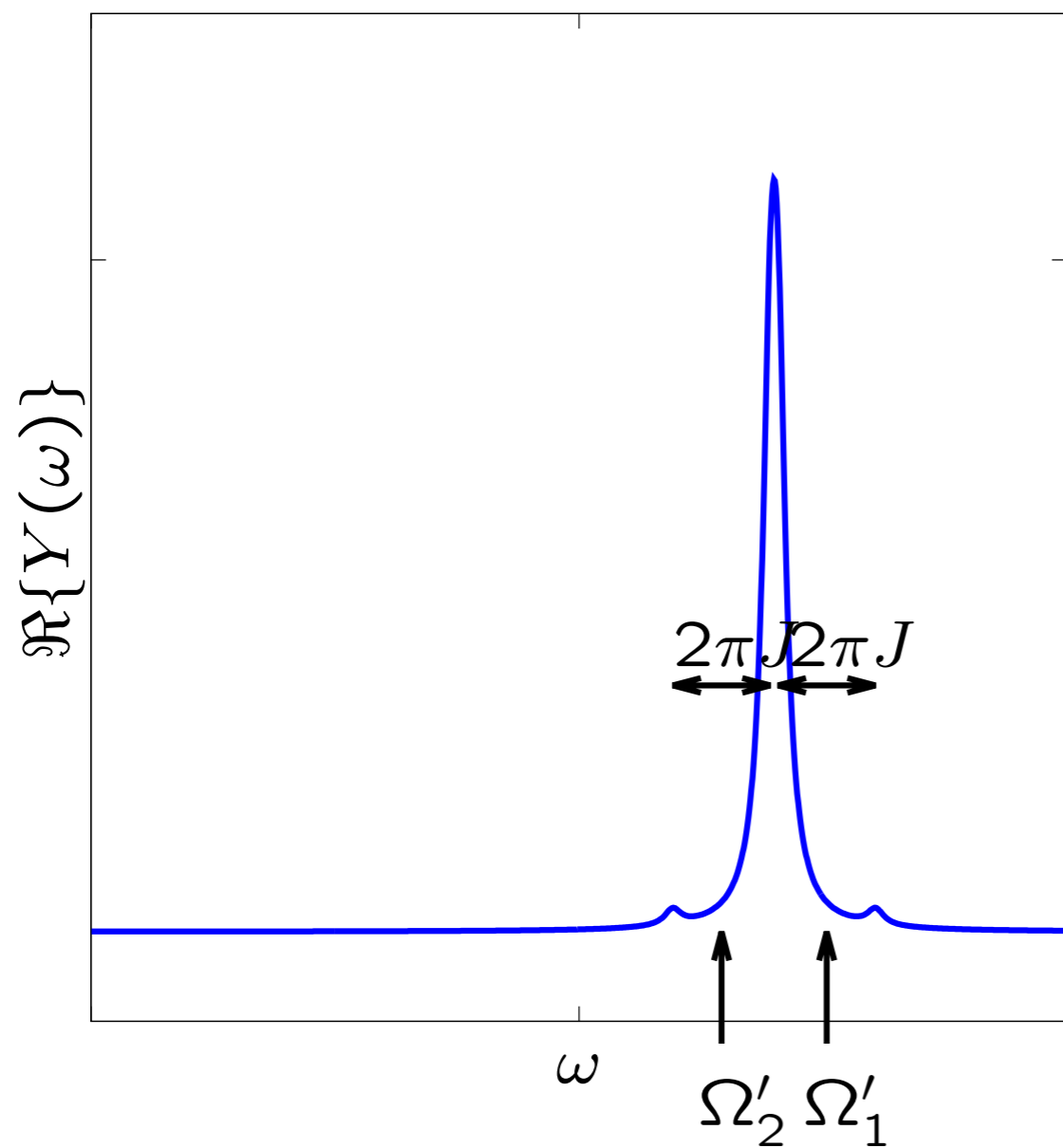
Spectrum of a strongly coupled pair

$$\Omega_1 - \Omega_2 = 2.0\pi J$$



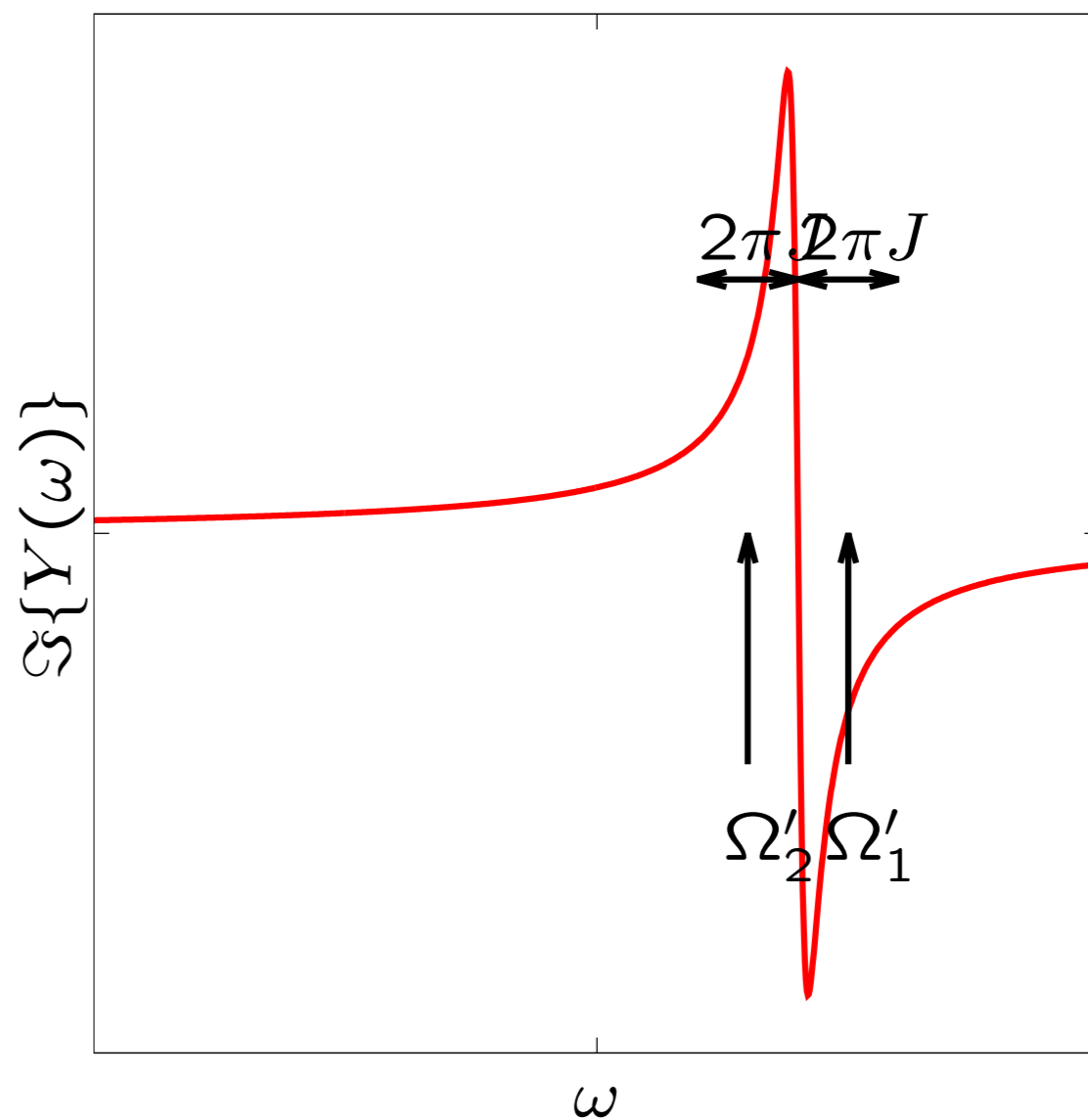
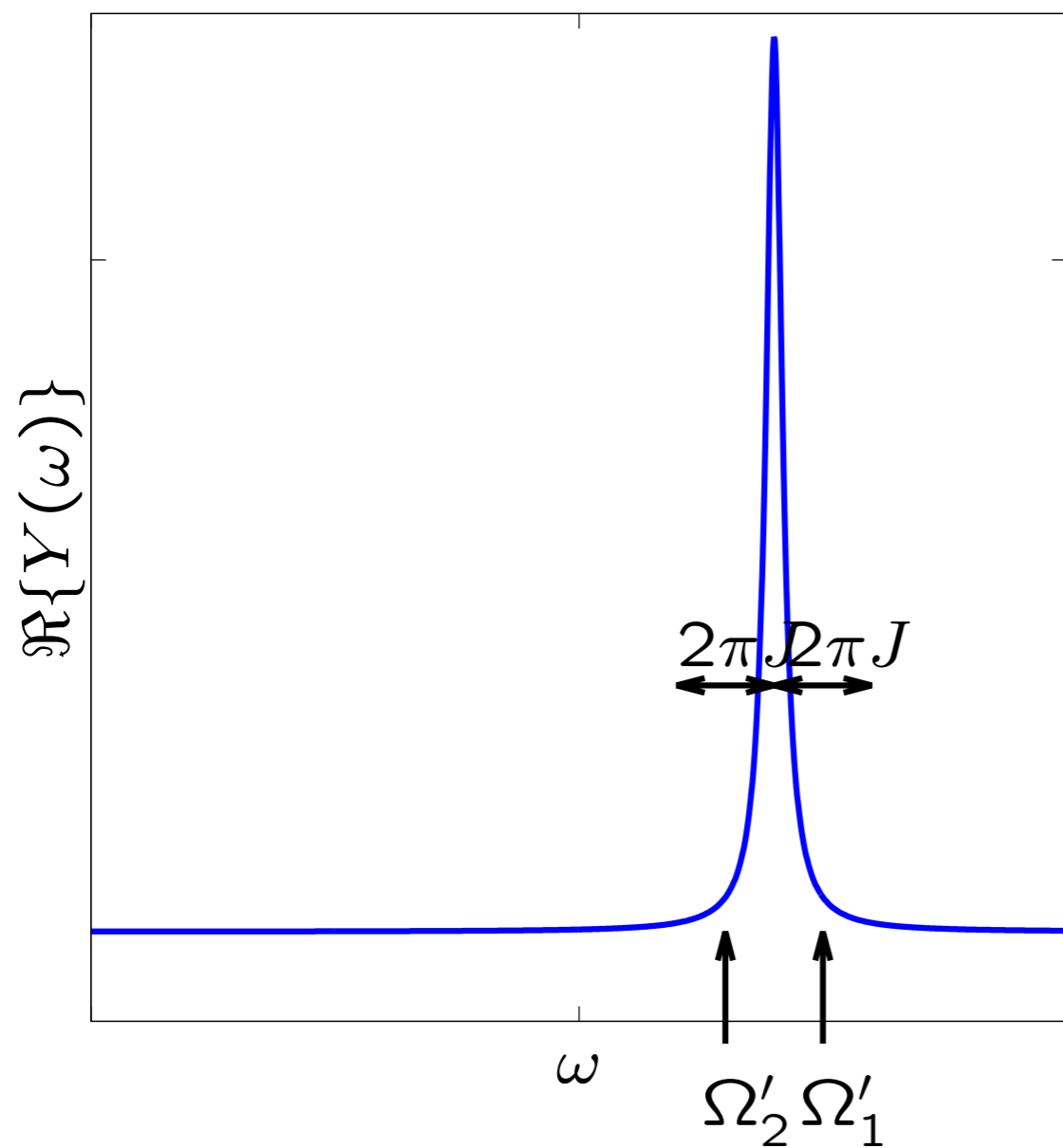
Spectrum of a strongly coupled pair

$$\Omega_1 - \Omega_2 = 0.8\pi J$$



Spectrum of a strongly coupled pair

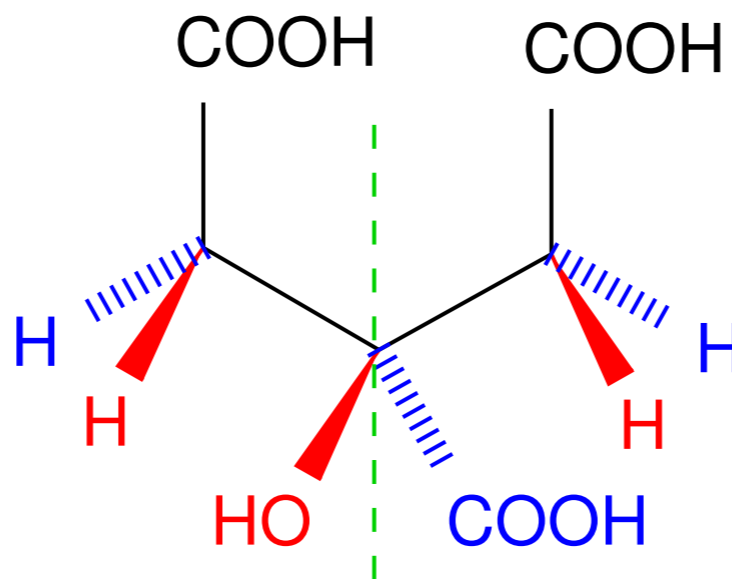
$$\Omega_1 - \Omega_2 = 0.0\pi J$$



Magnetic equivalence

- $\omega_{0,1} = \omega_{0,2}$ molecular symmetry or accident
- $J_{13} = J_{23}, \quad J_{14} = J_{24}, \dots$

Existence of a **plane of symmetry** is not sufficient, the plane must bisect the particular pair of nuclei:



H (closer to **OH**) and **H** (closer to **COOH**) not equivalent

Magnetic equivalence: eigenfunctions

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

stationary states (eigenfunctions of \mathcal{H}')

Magnetic equivalence: eigenvalues

$$\hat{I}^2 = (\hat{I}_1 + \hat{I}_2)^2 = \hat{I}_1^2 + \hat{I}_2^2 + 2\hat{I}_{1x}\hat{I}_{2x} + 2\hat{I}_{1y}\hat{I}_{2y} + 2\hat{I}_{1z}\hat{I}_{2z}$$

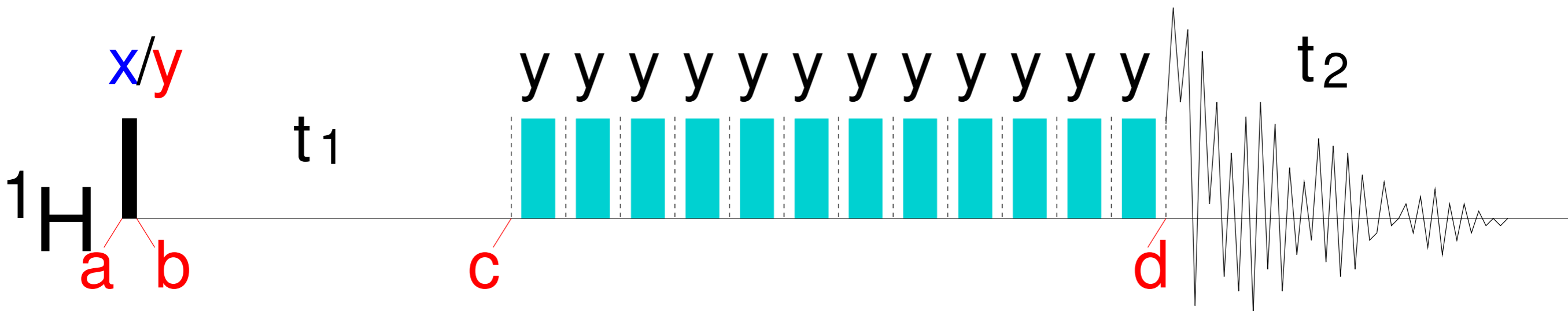
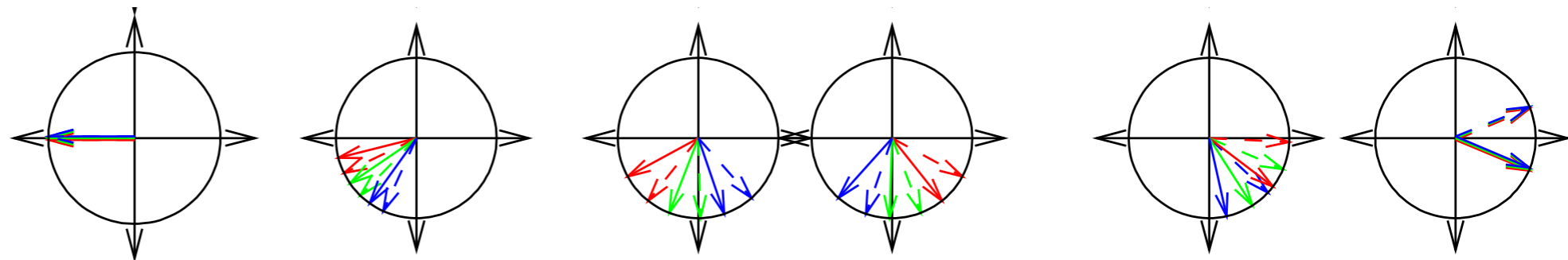
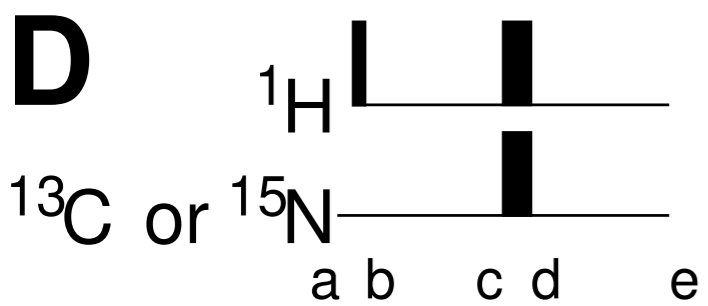
$$\mathcal{H}' = (\omega_0 + \pi J)\mathcal{I}_{1z} + (\omega_0 - \pi J)\mathcal{I}_{2z} + \pi J \cdot 2\mathcal{I}_{1z}\mathcal{I}_{2z}$$

Eigenfunction	\hat{I}_1^2	\hat{I}_2^2	\hat{I}^2	\hat{I}'_z	\mathcal{H}'
$ \alpha\rangle \otimes \alpha\rangle$	$3\hbar^2/4$	$3\hbar^2/4$	$2\hbar^2$	$+\hbar$	$+\omega_0 + \frac{\pi}{2}J$
$\frac{1}{\sqrt{2}} \alpha\rangle \otimes \beta\rangle + \frac{1}{\sqrt{2}} \beta\rangle \otimes \alpha\rangle$	$3\hbar^2/4$	$3\hbar^2/4$	$2\hbar^2$	0	$+\frac{\pi}{2}J$
$\frac{1}{\sqrt{2}} \alpha\rangle \otimes \beta\rangle - \frac{1}{\sqrt{2}} \beta\rangle \otimes \alpha\rangle$	$3\hbar^2/4$	$3\hbar^2/4$	0	0	$-\frac{3\pi}{2}J$
$ \beta\rangle \otimes \beta\rangle$	$3\hbar^2/4$	$3\hbar^2/4$	$2\hbar^2$	$-\hbar$	$-\omega_0 + \frac{\pi}{2}J$

TOCSY (TOtally Correlated SpectroscopY)

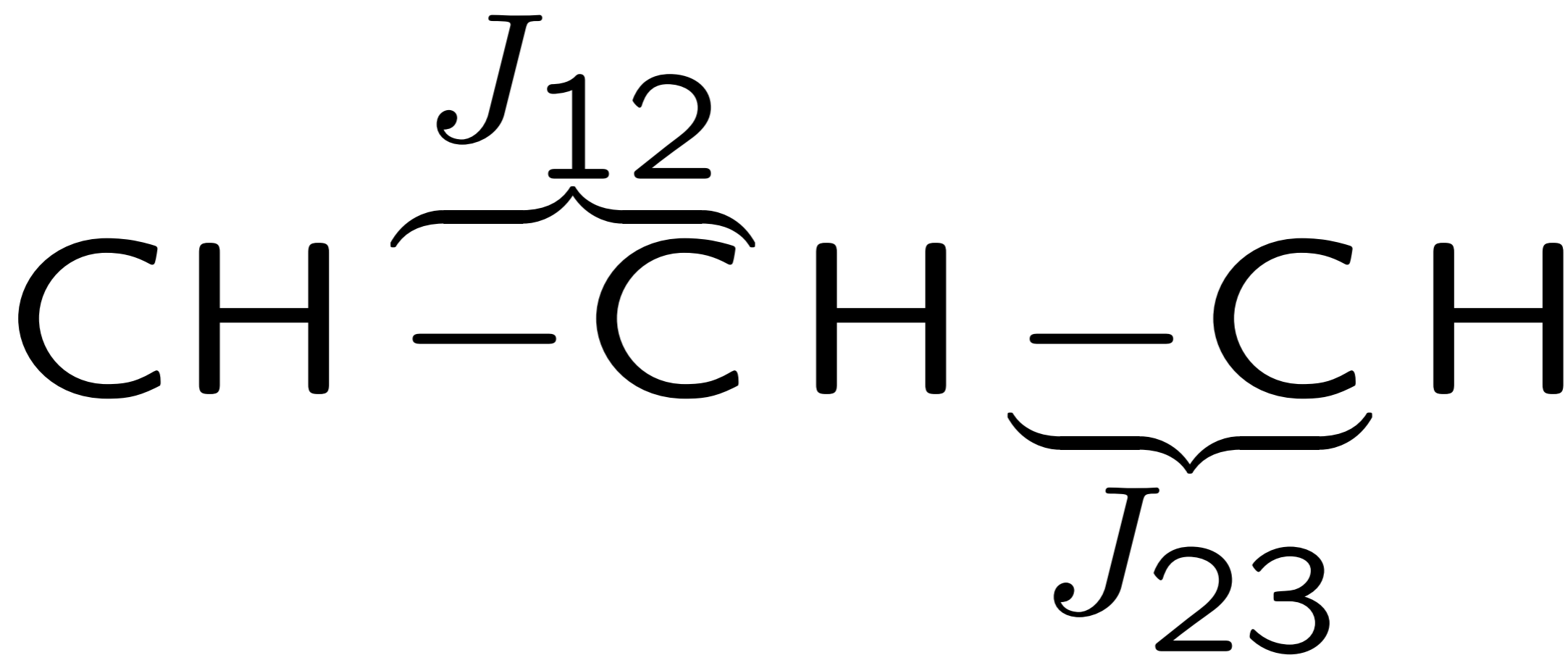
$$\gamma_1 = \gamma_2, \quad \omega_{0,1} = \omega_{0,2}, \quad \Omega_1 = \Omega_2$$

D

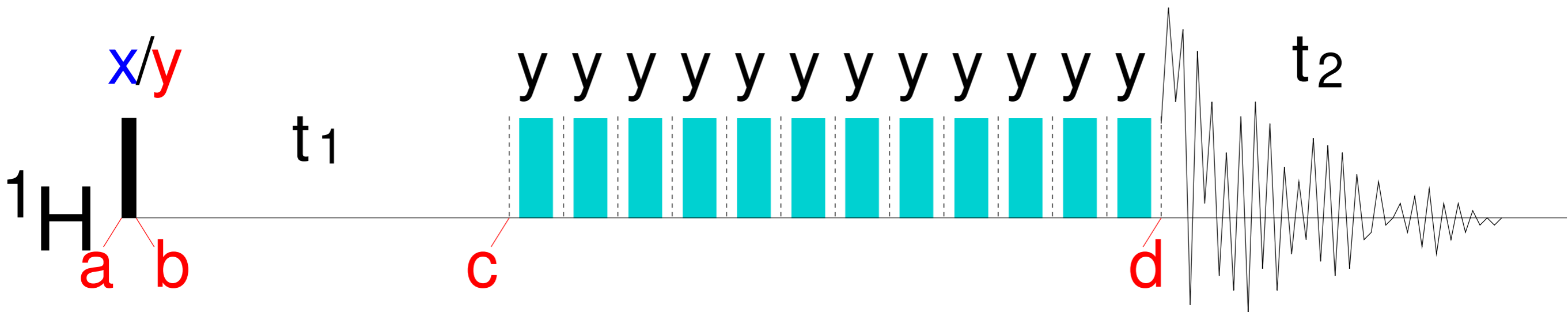


TOCSY

Simple example:

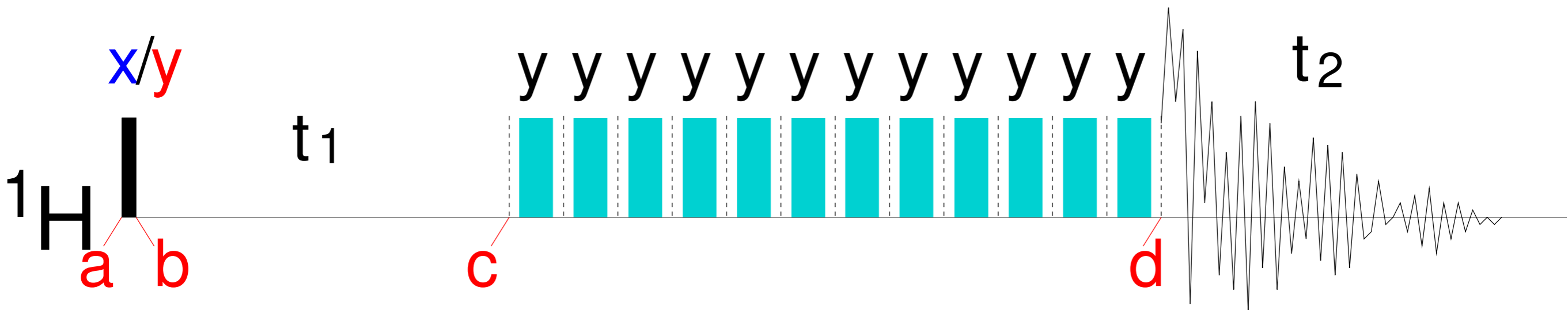


TOCSY



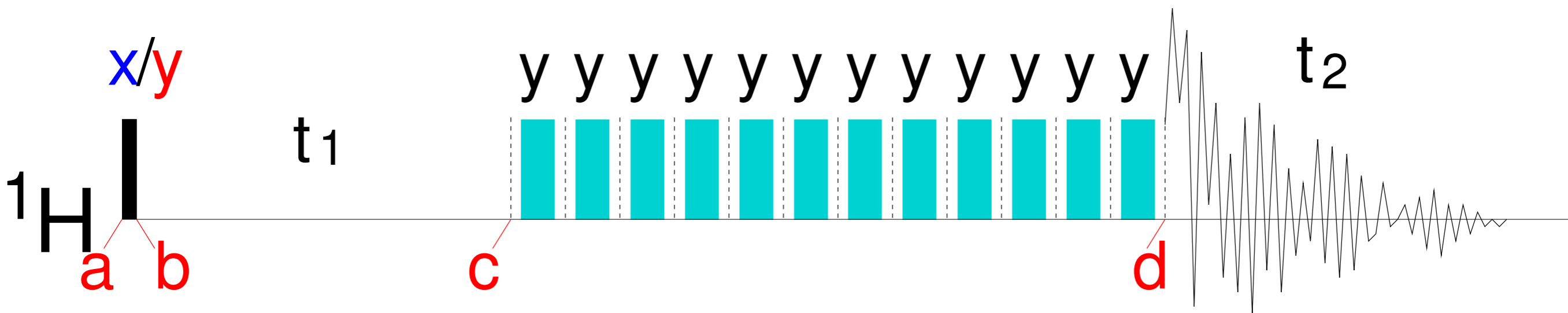
$$\hat{\rho}(a) = \frac{1}{4}(\mathcal{I}_t + \kappa \mathcal{I}_{1z} + \kappa \mathcal{I}_{2z} + \kappa \mathcal{I}_{3z})$$

TOCSY



$$\hat{\rho}(b) = \frac{1}{4}(\mathcal{I}_t - \kappa \mathcal{I}_{1y} - \kappa \mathcal{I}_{2y} - \kappa \mathcal{I}_{3y})$$

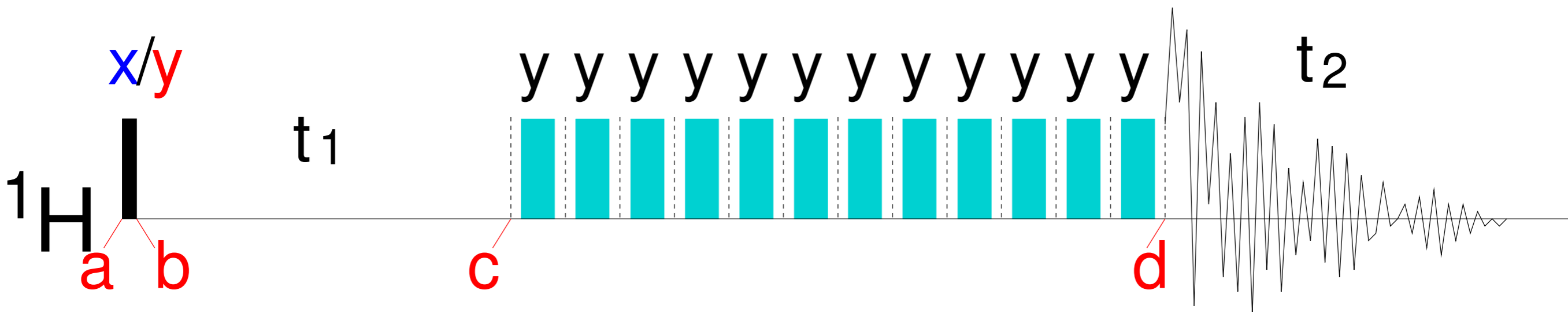
TOCSY



TOCSY pulse train applied with 90° (y) phases \Rightarrow

- $\mathcal{I}_{1y}, \mathcal{I}_{2y}, \mathcal{I}_{3y}$ components of the density matrix intact
- operators with \mathcal{I}_{nx} and \mathcal{I}_{nz} rotate "about" the \mathcal{I}_{ny} "axis"
- long rotation randomizes polarization in x and z
- only the $\mathcal{I}_{1y}, \mathcal{I}_{2y}, \mathcal{I}_{3y}$, "locked" in y , survive
- only evolution of $\mathcal{I}_{1y}, \mathcal{I}_{2y}, \mathcal{I}_{3y}$ can give a signal

TOCSY

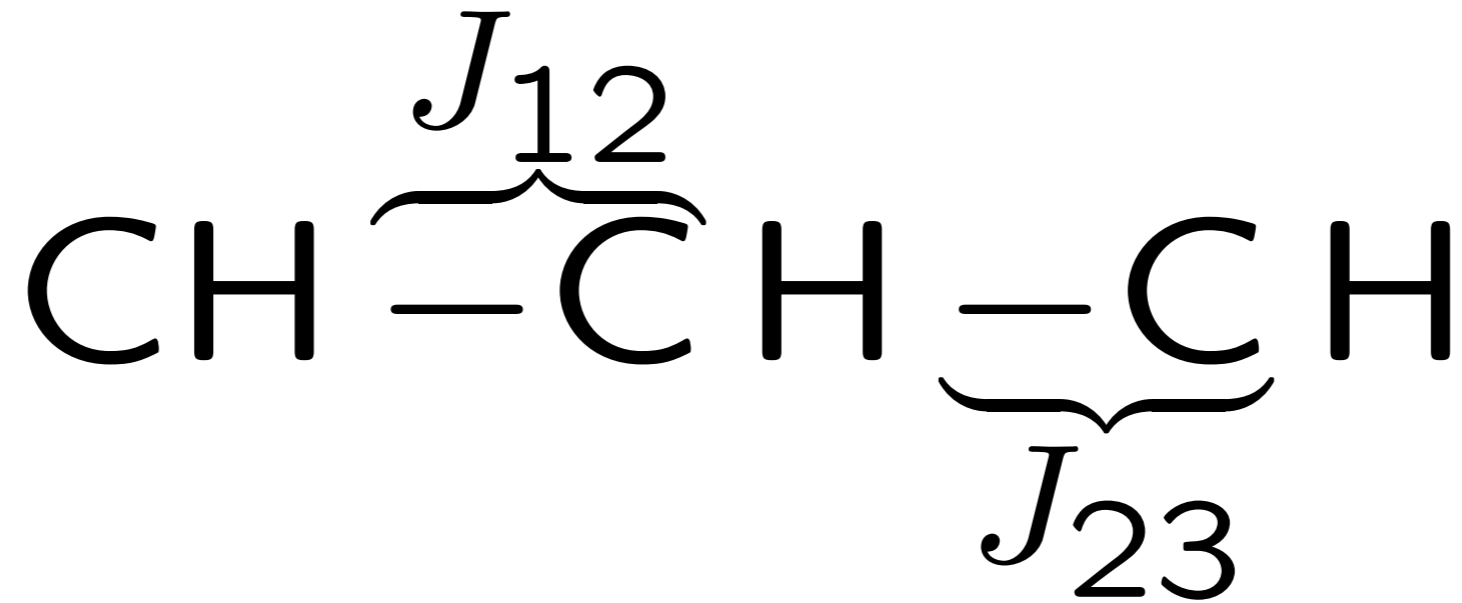


$$\hat{\rho}(c) = \dots - \frac{\kappa}{4} \cos(\Omega_1 t_1) \cos(\pi J_{12} t_1) \mathcal{I}_{1y}$$

$$- \frac{\kappa}{4} \cos(\Omega_2 t_1) \cos(\pi J_{12} t_1) \cos(\pi J_{23} t_1) \mathcal{I}_{2y}$$

$$- \frac{\kappa}{4} \cos(\Omega_3 t_1) \cos(\pi J_{23} t_1) \mathcal{I}_{3y}$$

TOCSY MIXING



$$\begin{aligned} \mathcal{H}_{\text{TOCSY}} = & \pi J_{12} (2\mathcal{I}_{1x}\mathcal{I}_{2x} + 2\mathcal{I}_{1y}\mathcal{I}_{2y} + 2\mathcal{I}_{1z}\mathcal{I}_{2z}) \\ & + \pi J_{23} (2\mathcal{I}_{2x}\mathcal{I}_{3x} + 2\mathcal{I}_{2y}\mathcal{I}_{3y} + 2\mathcal{I}_{2z}\mathcal{I}_{3z}) \end{aligned}$$

All components of $\mathcal{H}_{\text{TOCSY}}$ commute
their effects can be analyzed separately in any order

TOCSY MIXING

... but the analysis is not simple for > 2 nuclei

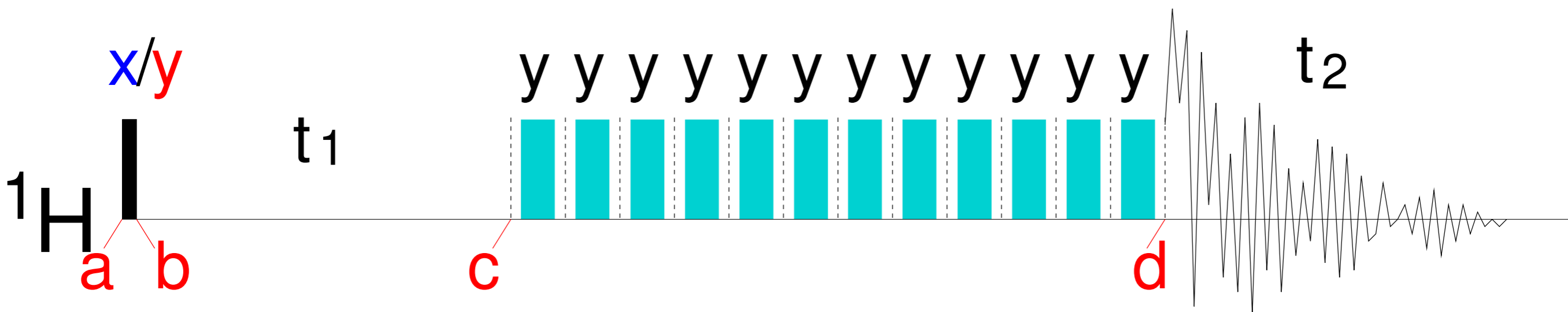
Commutator relations provide insight:

- $[\mathcal{I}_{1y}, \mathcal{H}_{\text{TOCSY}}] = -2i\pi J_{12}(\mathcal{I}_{1z}\mathcal{I}_{2x} - \mathcal{I}_{1x}\mathcal{I}_{2z}) \neq 0$
 \Rightarrow part of \mathcal{I}_{1y} is lost

- $[\mathcal{I}_{1y} + \mathcal{I}_{2y}, \mathcal{H}_{\text{TOCSY}}] = 2i\pi J_{23}(\mathcal{I}_{2x}\mathcal{I}_{3z} - \mathcal{I}_{2z}\mathcal{I}_{3x}) \neq 0$
 \Rightarrow the loss of \mathcal{I}_{1y} is not fully regained by \mathcal{I}_{2y}

- $[\mathcal{I}_{1y} + \mathcal{I}_{2y} + \mathcal{I}_{3y}, \mathcal{H}_{\text{TOCSY}}] = 0$
 \Rightarrow some \mathcal{I}_{3y} must be created to keep $\mathcal{I}_{1y} + \mathcal{I}_{2y} + \mathcal{I}_{3y}$ constant despite $J_{13} = 0!$

TOCSY



$$\hat{\rho}(d) =$$

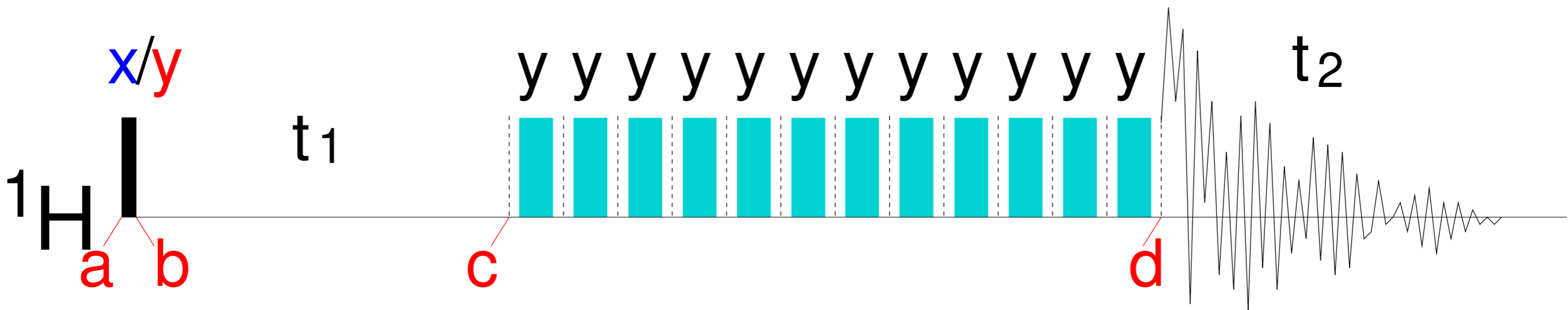
$$- \frac{\kappa}{4} \cos(\Omega_1 t_1) \cos(\pi J_{12} t_1) (a_{11} \mathcal{I}_{1y} + a_{12} \mathcal{I}_{2y} + a_{13} \mathcal{I}_{3y})$$

$$- \frac{\kappa}{4} \cos(\Omega_2 t_1) \cos(\pi J_{12} t_1) \cos(\pi J_{23} t_1)$$

$$\times (a_{21} \mathcal{I}_{1y} + a_{22} \mathcal{I}_{2y} + a_{23} \mathcal{I}_{3y})$$

$$- \frac{\kappa}{4} \cos(\Omega_3 t_1) \cos(\pi J_{23} t_1) (a_{31} \mathcal{I}_{1y} + a_{32} \mathcal{I}_{2y} + a_{33} \mathcal{I}_{3y})$$

TOCSY

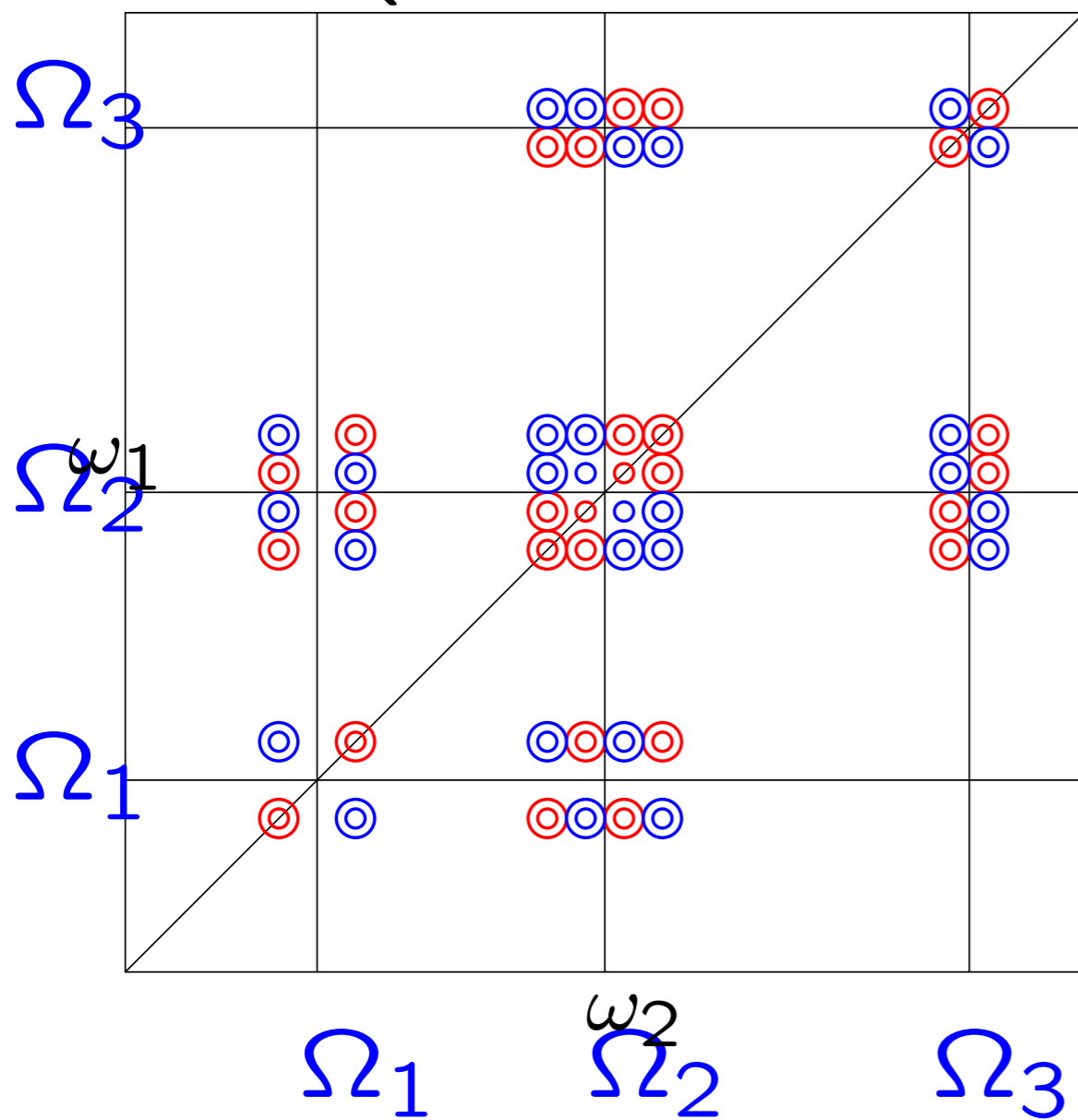


Evolution of $\hat{\rho}(t_2)$ analyzed as usually:

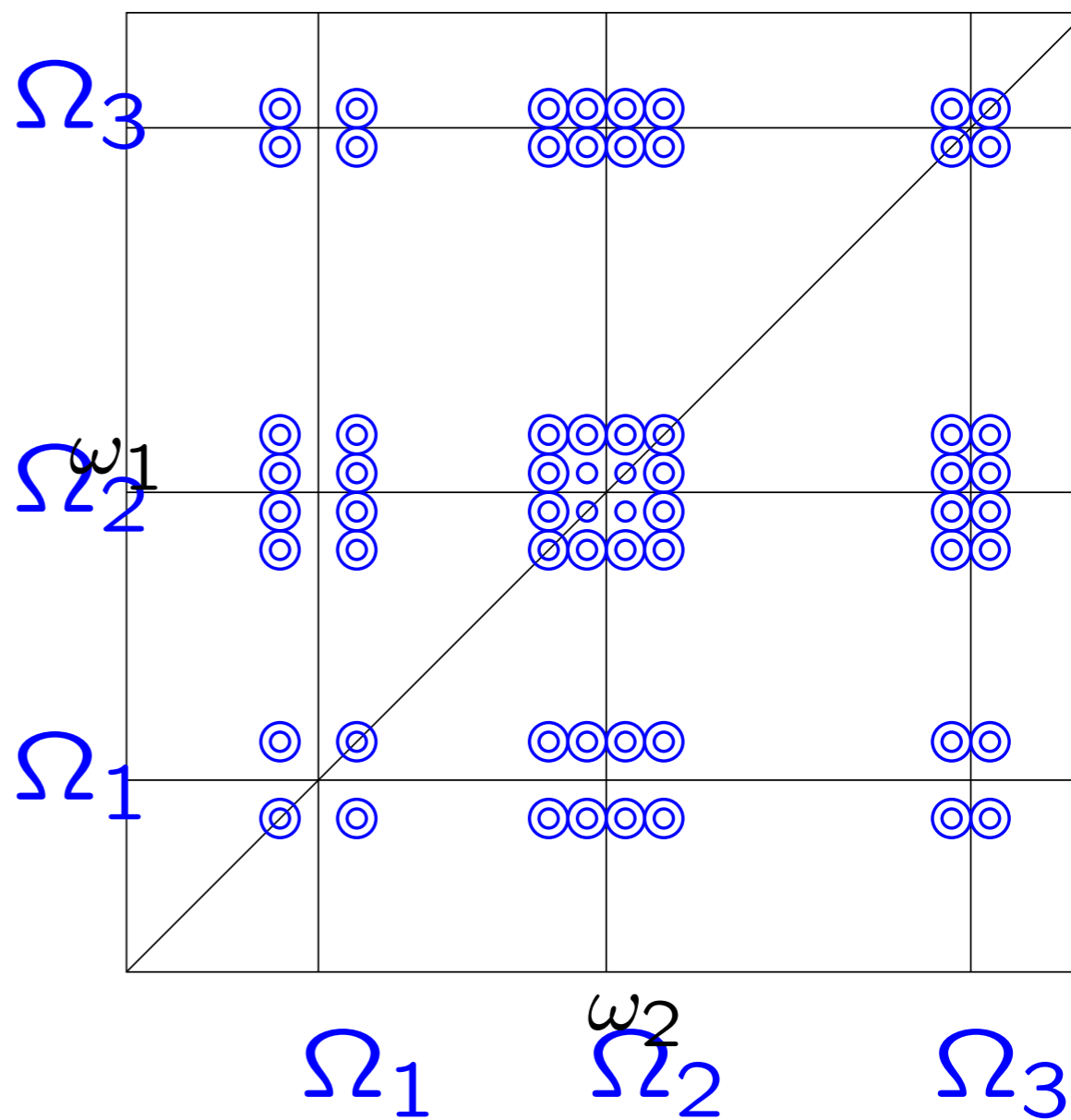
$$\begin{aligned} & \frac{\kappa a_{11}}{16} e^{-\bar{R}_2 t_1} \left(e^{-i(\Omega_1 - \pi J_{12}) t_1} + e^{-i(\Omega_1 + \pi J_{12}) t_1} \right) e^{-\bar{R}_2 t_2} \left(e^{-i(\Omega_1 - \pi J_{12}) t_2} + e^{-i(\Omega_1 + \pi J_{12}) t_2} \right) + \\ & \frac{\kappa a_{12}}{16} e^{-\bar{R}_2 t_1} \left(e^{-i(\Omega_1 - \pi J_{12}) t_1} + e^{-i(\Omega_1 + \pi J_{12}) t_1} \right) e^{-\bar{R}_2 t_2} \left(e^{-i(\Omega_2 - \pi J_{12}) t_2} + e^{-i(\Omega_2 + \pi J_{12}) t_2} \right) + \\ & \frac{\kappa a_{13}}{16} e^{-\bar{R}_2 t_1} \left(e^{-i(\Omega_1 - \pi J_{12}) t_1} + e^{-i(\Omega_1 + \pi J_{12}) t_1} \right) e^{-\bar{R}_2 t_2} \left(e^{-i(\Omega_3 - \pi J_{12}) t_2} + e^{-i(\Omega_3 + \pi J_{12}) t_2} \right) + \\ & \dots \end{aligned}$$

TOCSY spectrum

DQF-COSY



TOCSY



TOCSY vs. COSY

- different structural information
- TOCSY:
 - cross-peaks correlate all nuclei of a spin system
(spin system = network of J -coupled nuclei)
 - whole spin system in one spectrum
- COSY:
 - cross-peaks correlate only directly coupled nuclei
 - who is whose neighbor

HOMework:

Section 12.4.2

Strong coupling