

Figure 3.24 The half-life as a function of  $C_A$  for the data in Table 3.5 (with some extra points added).

that there are large differences between, for example, first- and second-order data. Further, if one takes data on a system that does not follow a simple rate equation, one knows it because the half-life plot is curved on a log-log scale. Still, the half-life method is difficult to automate. Consequently, half-life plots are now rarely seen in the literature.

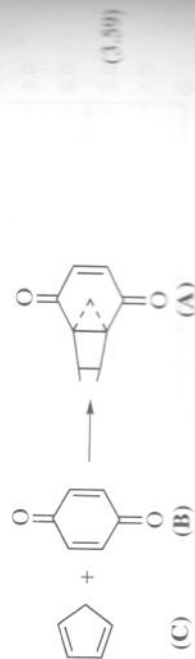
Equations (3.56) and (3.58) are very useful, however. One can use these equations to decide how to run experiments, or to estimate activation barriers from very little data. See Section 2.5.1 for more information.

There are other methods in the literature, including the Gugenheim method. These methods have largely disappeared as computers made their appearance. For an older review, see Roseveare (1931).

### 3.13 FITTING DATA TO EMPIRICAL RATE LAWS: MULTIPLE REACTANTS

The results in Section 3.7 are useful only in the case of a reaction where a single reactant is converted into products. In a more typical reaction, two reactants, for example, C and B, react to form products. One needs a more complex analysis to consider those cases. In the work that follows, we will derive an expression for the conversion versus time of a system with multiple reactions. However, we need some further information first.

For example, consider the Diels-Alder reaction of benzoquinone (B) and cyclopentadiene (C) to yield an adduct (A):



It is useful to consider what data we need to analyze rate data for reaction (3.59). Next, we will derive an expression for the conversion of benzoquinone as a function of time

assuming that we run the reaction in a constant-volume batch reactor and that the rate equation for the reaction is of the form

$$r_B = -k_B C_B C_C \quad (3.60)$$

where  $C_B$  is the benzoquinone concentration,  $C_C$  is the cyclopentadiene concentration,  $r_B$  is the rate of formation of benzoquinone and  $k_B$  is a constant. According to equation (3.31)

$$\tau = C_B^0 \int_0^{X_B} \frac{dX_B}{(-r_B)} \quad (3.61)$$

where  $X_B$  is the conversion, and  $\tau$  is the reaction time. Combining equations (3.60) and (3.61) yields

$$\tau = C_B^0 \int_0^{X_B} \frac{dX_B}{k_B C_B C_C} \quad (3.62)$$

In order to use equation (3.62), we need an expression for  $C_B$  and  $C_C$  in terms of  $X_B$ . In the next section, we will describe how to get it.

#### 3.13.1 The Stoichiometric Table

Consider a general reaction



where  $a$ ,  $b$ ,  $c$ , and  $d$  are the stoichiometric coefficients for species A, B, C, and D, respectively. Assume that  $N_A^0$  moles of A and  $N_B^0$  moles of B are loaded into a closed isothermal vessel. Next, we will derive a relation between the concentrations of all of the species as a function of the concentration of a single species, A. To simplify the algebra, it is useful to work in terms of a new variable,  $X_A$ , the fractional conversion of A, where  $X_A$  is defined as

$$X_A = \frac{N_A^0 - N_A}{N_A^0} \quad (3.64)$$

Physically  $X_A$  is the fraction of the reactant A that has been converted into products. One should memorize this definition before proceeding.

Next, it is useful to calculate the concentrations of each of the species in the reactor as a function of  $X_A$ . Fogler [1998] has a neat trick that makes it easier to keep track of this. The trick is called a **stoichiometric table**.

The stoichiometric table is a table where the following information is presented (for a batch system):

- Column 1: The particular species
- Column 2: The number of moles of each species initially present
- Column 3: The change in the number of moles due to reaction
- Column 4: The number of moles remaining in the reactor after conversion  $X_A$

Table 3.7 is a stoichiometric table for reaction (3.63) with some of the information missing. Next, let's calculate the changes in moles of each of the species as a function of  $X_A$ . Assume that we had  $N_A^0$  moles of A and  $N_B^0$  moles of B in the reactor. Then we can

**Table 3.7** The start of a stoichiometric table for reaction (3.63)

Species	Initial Moles	Change in Moles	Final Moles
A	$N_A^0$		$N_A^f$
B	$N_B^0$		$N_B^f$
C	$N_C^0$		$N_C^f$
D	$N_D^0$		$N_D^f$
Inerts	$N_I^0$		$N_I^f$
Total	$N_T^0 = N_A^0 + N_B^0 + N_C^0 + N_D^0 + N_I^0$		$N_T^f$

the reaction for a time,  $t$ , so that the conversion of species A is  $X_A$ . From the definition of the conversion we obtain

$$\text{Moles of A converted} = N_A^0 X_A \quad (3.65)$$

From the stoichiometry of the reaction, the number of moles of B converted is given by

$$\text{Moles of B converted} = \left(\frac{b}{a}\right) N_A^0 X_A \quad (3.66)$$

Similarly:

$$\text{Moles of C formed} = \left(\frac{c}{a}\right) N_A^0 X_A \quad (3.67)$$

$$\text{Moles of D formed} = \left(\frac{d}{a}\right) N_A^0 X_A \quad (3.68)$$

It is important to keep track of the sign. During the reaction, the concentrations of A and B decrease while the concentrations of C and D rise. Consequently, the change in moles of B is:

$$\Delta N_B = -\left(\frac{b}{a}\right) X_A N_A \quad (3.69)$$

while the change in moles of C is

$$\Delta N_C = +\left(\frac{c}{a}\right) X_A N_A \quad (3.70)$$

Therefore, the stoichiometric table is as given in Table 3.8:  
One can slightly simplify Table 3.8 by noting

$$-a - b + c + d = \sum \nu_i = \Delta n \quad (3.71)$$

where  $\Delta n$  is the change in moles when reaction (3.63) goes one time. Substituting equation (3.67) into Table 3.8 yields Table 3.9.

### 3.13.2 Derivation of the Performance Equation for Reaction (3.59)

Now it is useful to go back and derive a performance equation for reaction (3.59). Let's go back to equation (3.62). Equation (3.62) was a performance equation for reaction (3.59),

**Table 3.8** The stoichiometric table for reaction (3.63)

Species	Initial Moles	Change in Moles	Final Moles
A	$N_A^0$	$-N_A^0 X_A$	$N_A^0(1 - X_A)$
B	$N_B^0$	$-\left(\frac{b}{a}\right) N_A^0 X_A$	$N_B^0 - \left(\frac{b}{a}\right) N_A^0 X_A$
C	$N_C^0$	$+\left(\frac{c}{a}\right) N_A^0 X_A$	$N_C^0 + \frac{c}{a} N_A^0 X_A$
D	$N_D^0$	$+\left(\frac{d}{a}\right) N_A^0 X_A$	$N_D^0 + \frac{d}{a} N_A^0 X_A$
Inerts	$N_I^0$	0	$N_I^0$
Total	$N_T^0$	$\left(\frac{-a - b + c + d}{a}\right) N_A^0 X_A$	$N_T^0 + N_A^0 \left(\frac{-a - b + c + d}{a}\right) X_A$

**Table 3.9** An alternative stoichiometric table for reaction (3.63)

Species	Initial Moles	Change in Moles	Final Moles
A	$N_A^0$	$-N_A^0 X_A$	$N_A^0(1 - X_A)$
B	$N_B^0$	$-\left(\frac{b}{a}\right) N_A^0 X_A$	$N_B^0 - \left(\frac{b}{a}\right) N_A^0 X_A$
C	$N_C^0$	$+\left(\frac{c}{a}\right) N_A^0 X_A$	$N_C^0 + \frac{c}{a} N_A^0 X_A$
D	$N_D^0$	$+\left(\frac{d}{a}\right) N_A^0 X_A$	$N_D^0 + \frac{d}{a} N_A^0 X_A$
Inerts	$N_I^0$	0	$N_I^0$
Total	$N_T^0$	$+N_A^0 \left(\frac{\Delta n}{a}\right) X_A$	$N_T^0 + N_A^0 \left(\frac{\Delta n}{a}\right) X_A$

However, we needed an expression for  $C_C$  as a function of  $C_B$  so that we can integrate the equation. We can get the needed expression from a stoichiometric table. Table 3.10 shows a general stoichiometric table, while Table 3.11 shows the specific stoichiometric table for the case considered here.

Substituting results from Table 3.11 into equation (3.60) yields

$$(-r_B) = K_B(C_B)(C_C) = K_B(C_B^0(1 - X_B))(C_C^0 - X_B C_B^0) \quad (3.72)$$

According to equation (3.30)

$$\tau = C_B^0 \int_0^{X_B} \frac{dX_B}{(-r_B)} \quad (3.73)$$

**Table 3.10** The stoichiometric table for the reaction  $bB + cC \Rightarrow aA$

Species	Initial Concentration	Change	Final Concentration
B	$C_B^0$	$-C_B^0 X_B$	$C_B^0(1 - X_B)$
C	$C_C^0$	$-(c/b)C_B^0 X_B$	$C_C^0 - (c/b)C_B^0 X_B$
A	$C_A^0$	$(a/b)C_B^0 X_B$	$C_A^0 + (a/b)C_B^0 X_B$

Table 3.11 The stoichiometric table for the reaction  $C + B \Rightarrow A$ 

Species	Initial Concentration	Change	Final Concentration
B	$C_B^0$	$-C_B^0 X_B$	$C_B^0(1 - X_B)$
C	$C_C^0$	$-C_B^0 X_B$	$C_C^0 - C_B^0 X_B$
A	$C_A^0$	$C_B^0 X_B$	$C_A^0 + C_B^0 X_B$

Substituting equation (3.72) into equation (3.73) yields

$$\tau = C_B^0 \int_0^{X_B} \frac{dX_B}{(k_B C_B^0(1 - X_B)(C_C^0 - X_B C_B^0))} \quad (3.74)$$

The integral equation (3.74) looks imposing. Fortunately, I was able to look it up in Gradshteyn and Ryzhik (1965).

$$\tau = \frac{\left[ \frac{C_B^0}{C_C^0} \left( \frac{C_C^0 - X_B}{1 - X_B} \right) \right]}{k_B(C_C^0 - C_B^0)} \quad (3.75)$$

Equation (3.75) has two interesting limits: the limit where  $C_C^0 \gg C_B^0$  and the limit where  $C_C^0 = C_B^0$ . The first limit, called the "swamping limit," corresponds to running the reaction with a huge excess of cyclopentadiene (C). If  $C_C^0$  is in large excess,  $C_C^0/C_B^0 \gg 1 \geq X_B$ . Therefore, the  $X_B$  term in the numerator of the log term in equation (3.75) will be negligible. Similarly,  $C_B^0$  in the denominator of equation (3.75) will be negligible. Consequently, when there is a large excess of cyclopentadiene, equation (3.75) reduces to

$$\tau = \frac{1}{k_B C_C^0} \ln \left( \frac{1}{1 - X_B} \right) \quad (3.76)$$

A comparison of equations (3.76) and (3.39) shows that equation (3.76) is a first-order rate equation with

$$k_1 = k_B C_C^0 \quad (3.77)$$

Physically, what is happening is that when  $C_C^0 \gg C_B^0$ , the concentration of cyclopentadiene does not change significantly during the reaction. The  $C_C$  is constant in the rate equation. Therefore, the reaction appears as if it were first-order.

The other key limit is when  $C_B^0 = C_C^0$ . Note that during reaction (3.59), benzoquinone (B) and cyclopentadiene (C) are used up at the same rate. Consequently, if initially  $C_B^0 = C_C^0$ , then

$$C_B = C_C \quad (3.78)$$

everywhere during the reaction. Consequently, the rate equation for reaction (3.59) will reduce to the equation for a second-order reaction.

One can derive the same result by letting  $C_B^0$  approach  $C_C^0$  in equation (3.84), and using l'Hôpital's rule to do the limit. The result is

$$\tau = \frac{1}{k_B C_B^0} \left( \frac{C_B^0}{C_B} - 1 \right) \quad (3.79)$$

Equation (3.79) is equivalent to equation (3.42), with  $n = 2$ . Equations (3.75) and (3.79) are used to plan experiments. First, one swamps the reactor with one species, say, C, and measures the concentration of B as a function of time, and uses Essen's method or Powell's method to see if the reaction follows equation (3.76). One then runs the reaction with equal B and C concentrations, and uses Powell's method or Essen's method to see if equation (3.79) works. If both equations fit the data, then one can be assured that the rate data follows equation (3.60). If either equation (3.76) or (3.79) fails, one needs a more complex procedure to fit the data. Such procedures are beyond the scope of the discussion here.

### 3.14 SEQUENTIAL REACTIONS

There is one other example that we will need to consider later in this book, which is the limit where a reactant A is first converted into an intermediate, I, and then into a product, P:



Next, we will derive equations for the concentrations of A, B, and C versus  $\tau$  assuming that reactions 1 and 2 are both first-order.

If reaction 1 is first-order, then

$$\frac{dC_A}{d\tau} = r_A = -k_1 C_A \quad (3.81)$$

where  $C_A$  is the concentration of A. Solving equation (3.81) yields

$$C_A = C_A^0 e^{-k_1 \tau} \quad (3.82)$$

If reactions 1 and 2 are first-order, then

$$\frac{dC_I}{dt} = k_1 C_A - k_2 C_I \quad (3.83)$$

where  $C_I$  is the concentration of I.

Combining equations (3.82) and (3.83) and integrating yields

$$C_I = \frac{k_1 C_A^0}{k_2 - k_1} (e^{-k_1 \tau} - e^{-k_2 \tau}) + C_I^0 e^{-k_2 \tau} \quad (3.84)$$

where  $C_1^0$  is the initial concentration of B. We could integrate and calculate the concentration of P. However, it is much easier to calculate the concentration of P from a mass balance.

$$C_P = C_A^0 + C_1^0 + C_P^0 - C_A C_1 \quad (3.85)$$

Equations (3.82)–(3.84) are the key equations for sequential reactions.

Figure 3.25 shows a plot of equation  $C_A$ ,  $C_1$ , and  $C_P$  versus  $k_1\tau$  for various values of  $k_2/k_1$  calculated from equations (3.82), (3.84), and (3.85). We have plotted the data as a function of  $k_1\tau$ , to eliminate the  $k_1$  dependence.

The concentration of the reactant A decreases exponentially with time as was shown in Figure 3.14. However, in Figure 3.26, we plot the data as a function of  $k_1\tau$ . All of the  $C_A$  data fall on a universal curve, independent of  $k_1$  and  $k_2$ .

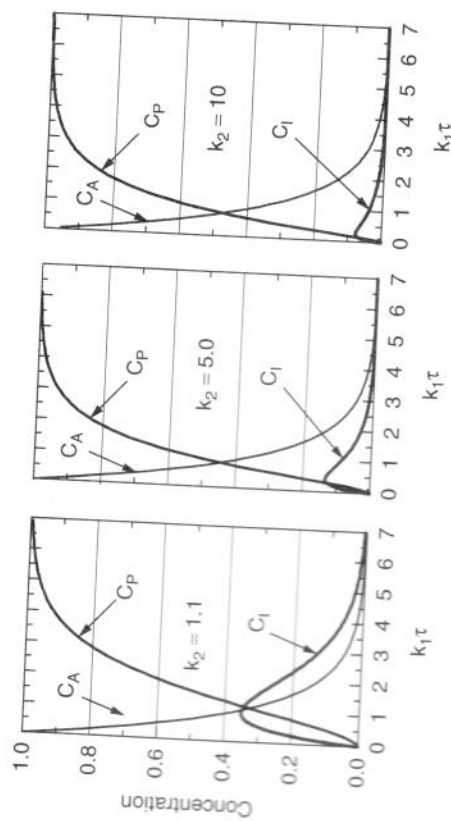


Figure 3.25 A plot of equations (3.82)–(3.85) with  $C_A^0 = 1$ ,  $C_1^0 = 0$ .

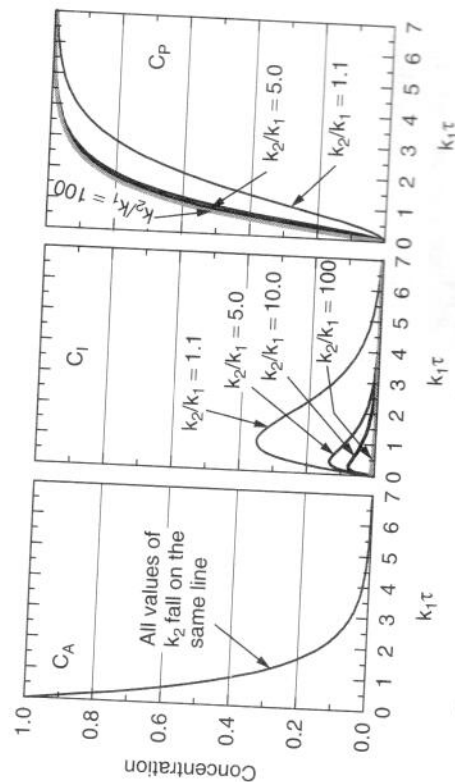


Figure 3.26 A plot of  $C_1$  and  $C_A$  versus  $k_1\tau$  for  $k_2/k_1 = 1, 5, 10, 100$ .

The concentration of the intermediate shows quite different behavior. The intermediate concentration increases, reaches a maximum, and then declines again. The height of the maximum decreases as  $k_2/k_1$  increases. Physically, what is happening is that in the initial part of the reaction, the reactant A is being converted into the intermediate, so initially the concentration of the intermediate rises. However, as the concentration of the intermediate builds up, the rate of conversion of the intermediate into the product P begins to become important. Eventually, the rate of destruction of the intermediate through reaction 2 gets to be larger than the rate of production of the intermediate through reaction 1. At that stage, the concentration of the intermediate falls again.

For future reference, it is useful to plot two other quantities:

$$C_P^s = C_A^0 - C_A \quad (3.86)$$

and

$$C_1^s = \frac{k_1}{k_2} C_A \quad (3.87)$$

Notice that equation (3.85) reduces to equation (3.86) when  $C_1 = C_1^0 = 0$ . Therefore,  $C_P^s$  in equation (3.86) will be the product concentration if the intermediate concentration is negligible.  $C_1^s$  is more complicated.  $C_1^s$  is the concentration of the intermediate that one would calculate if the derivative on the left side of equation (3.82) were zero.

Figure 3.27 shows a plot of the  $C_P^s$  and  $C_1^s$  versus  $k_1\tau$ . We have multiplied  $C_1^s$  and  $C_1$  by  $k_2/k_1$  to make the figure easier to see. Notice that when  $k_2/k_1 > 10$ ,  $C_1^s$  is barely distinguishable from  $C_1$ , except at the very start of the reaction.  $C_P^s$  is barely distinguishable from  $C_P$ . That will be very important for the discussion in the next

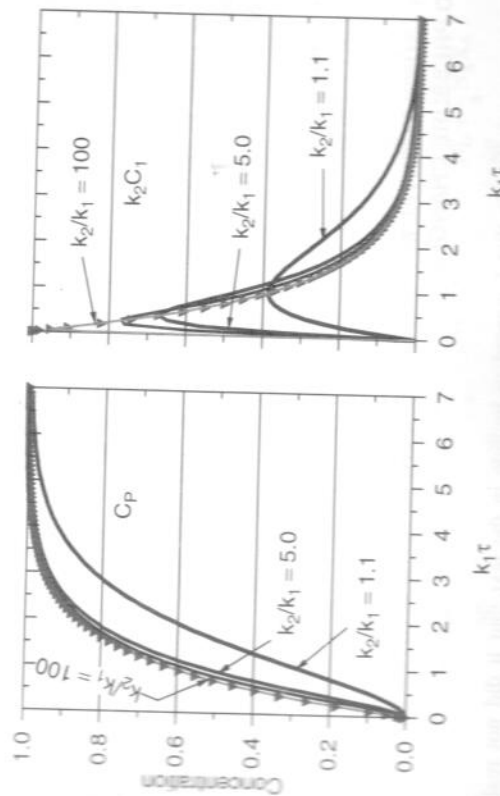


Figure 3.27 Left — a plot of  $C_P$  (lines) and  $C_P^s$  (triangles) versus  $k_1\tau$ . Right — a plot of  $k_2C_1$  (lines) and  $k_2C_1^s$  (points) versus  $k_1\tau$ .

**Table 3.12 Summary of key concepts**

Two methods to measure rates: direct and indirect	Methods to analyze direct data
Direct: high accuracy; many runs needed	Least squares
Indirect: lower accuracy; fewer experiments	Nonlinear least-squares
Methods to analyze indirect data	Nonlinear least-squares easier and more accurate
Essen	Key equations for indirect data
Construct plots of $\ln(C_A^0/C_A)$ , $(C_A^0/C_A)^n - 1$	$N_A^0 \int_0^{X_A} \frac{dX_A}{V(-r_A)} = \tau \quad (3.26)$
See if linear	$C_A^0 \int_0^{X_A} \frac{dX_A}{-r_A} = \tau \quad (3.28)$
Van't Hoff	$\int_{C_A^f}^{C_A^0} \frac{dC_A}{-r_A} = \tau \quad (3.31)$
Calculate $k_1$ , $k_2$ , $k_3$	$\frac{1}{k_1} \ln \left( \frac{C_A^0}{C_A} \right) = \tau \quad (3.39)$
See if constant	$\frac{1}{(n-1)k_n} \left[ \left( \frac{C_A^0}{C_A} \right)^{n-1} - 1 \right] = \tau \quad (3.42)$
Van't Hoff—easier and more accurate	

chapter, because it means that one can calculate an accurate value of the concentration of the intermediate by assuming the derivative in equation (3.82) to be zero whenever  $k_2/k_1 > 10$ .

### 3.15 SUMMARY

Table 3.12 summarizes the key results from this chapter. In this chapter, we reviewed some basic kinetic analysis. We discussed the order of a reaction and the activation energy, and showed how those quantities could be measured. We then briefly discussed planning experiments and noted that a key decision in planning an experiment is to decide between direct and indirect methods. Direct methods require harder experiments, but the data are easier to analyze. Indirect methods require easy experiments, but the analysis of the rate data are less certain. We also discussed how one analyzes data from batch reactors. We expect that most of these concepts are already familiar to our readers. Still, we recommend that the reader review the solved examples at the end of this chapter before going on to Chapter 4.

### 3.16 SUPPLEMENTAL MATERIAL: MEASURING RATE DATA FROM VOLUME AND PRESSURE CHANGES

There is one other topic that I wanted to include in the text, but it did not really fit, which is to discuss how one uses pressure and volume changes to infer rate data. As noted previously, kinetic measurements first became popular during the latter half of

the nineteenth century. At the time, it was hard to make kinetic measurements. None of the spectroscopic techniques described in Chapter 1 had been invented. Accurate microbalances were not generally available. Analytic techniques were primitive. As a result, people needed to find a series of tricks to measure the concentration changes. A common experiment was to run a gas-phase reaction in a closed vessel and measure the change in the pressure of the gas as a function of time.

For example, in 1883, Van't Hoff examined the decomposition of arsine, specifically, reaction (3.5), on a glass vessel. Today, one would examine that reaction with a microbalance as described in Section 3.5. In 1883, however, no one had a balance that was sensitive enough to measure how much arsenic was deposited. Van't Hoff had to find another method to make the measurements. He loaded the arsine into a closed flask and measured the pressure as a function of time. Note that reaction (3.5) converts 2 mol of arsine gas to 3 mol of hydrogen gas, plus 1 mol of solid arsenic. If 2 mol are converted into 3 mol in a fixed-size vessel, the pressure in the vessel will go up.

Van't Hoff noted that if one measures the pressure as a function of time, one can back-calculate the amount of arsine that was converted and therefore calculate a rate. It is unusual to see this type of measurement today. However, there are many examples of this type in the literature. Consequently, it is useful to derive a series of equations to see how one can use the pressure in a system to calculate the concentrations of all of the species and the amount of gas reactant that is converted.

The derivation will start with the stoichiometric table we derived previously for the reaction



where  $-a$ ,  $-b$ ,  $c$ , and  $d$  are the stoichiometric coefficients for species A, B, C, and D, respectively. Assume that  $N_A^0$  moles of B are loaded into a closed isothermal vessel. Next, we will derive a relation between the number of moles of A in the vessel and the pressure in the vessel.

Recall from the discussion in Section 3.13.1 that the stoichiometric table for this reaction is given by a table such as Table 3.13 (see also Table 3.7).

Next, we will calculate the pressure in the system as a function of the initial pressure. According to the ideal-gas law:

$$P_T V = N_T RT \quad (3.89)$$

**Table 3.13 A stoichiometric table for reaction (3.87)**

Species	Initial Moles	Change in Moles	Final Moles
A	$N_A^0$	$-N_A^0 X_A$	$N_A^0(1 - X_A)$
B	$N_B^0$	$-\left(\frac{b}{a}\right) N_A^0 X_A$	$N_B^0 - \left(\frac{b}{a}\right) N_A^0 X_A$
C	$N_C^0$	$+\left(\frac{c}{a}\right) N_A^0 X_A$	$N_C^0 + \frac{c}{a} N_A^0 X_A$
D	$N_D^0$	$+\left(\frac{d}{a}\right) N_A^0 X_A$	$N_D^0 + \frac{d}{a} N_A^0 X_A$
Inerts	$N_I^0$	0	$N_I^0$
Total	$N_T^0$	$+N_A^0 \left(\frac{\Delta \text{mol}}{a}\right) X_A$	$N_T^0 + N_A^0 \left(\frac{\Delta \text{mol}}{a}\right) X_A$

where  $P_T$  is the total pressure of the reactor,  $V$  is the volume in the reactor,  $N_T$  is the number of moles in the reactor,  $R$  is the gas law constant, and  $T$  is the temperature. At the start of the reaction

$$P_T^0 V = N_T^0 RT \quad (3.90)$$

where  $P_T^0$  is the initial pressure of the reactor and  $N_T^0$  is the total number of moles in the reactor at the start of the reactions. Dividing equation (3.89) by equation (3.90) and rearranging yields

$$P_T = P_T^0 \left( \frac{N_T}{N_T^0} \right) \quad (3.91)$$

Substituting  $N_T$  and  $N_T^0$  from the stoichiometric table into equation (3.91) and rearranging yields

$$\frac{P_T}{P_T^0} = 1 + \left( \frac{N_A^0}{N_T^0} \right) \left( \frac{\Delta \text{mol}}{a} \right) X_A \quad (3.92)$$

Therefore, if one knows the pressure in a vessel as a function of time, one can calculate the conversion as a function of time.

Figure 3.28 shows a plot of the pressure calculated from equation (3.92) as a function of conversion for various values of the  $\Delta \text{mol}$ . Notice that the pressure always varies linearly with conversion. The pressure goes up when  $\Delta \text{mol}$  is positive, while the pressure goes down when  $\Delta \text{mol}$  is negative. Consequently, whenever  $\Delta \text{mol}$  is nonzero, one can use the pressure to estimate the conversion.

Examples 3.B and 3.C illustrate the use of the stoichiometric table to calculate the conversion as a function of time.

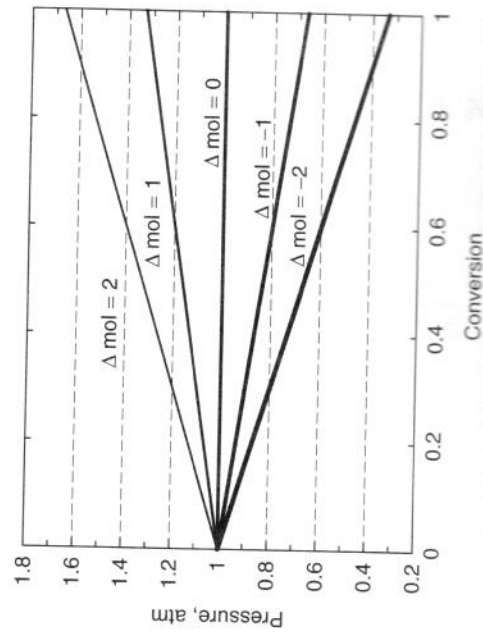


Figure 3.28 A plot of the pressure versus conversion calculated from equation (3.92) with  $N_A^0 = 1$  mol,  $N_T^0 = 3$  mol, and  $a = 1.0$ .

### 3.17 SOLVED EXAMPLES

**Example 3.A Fitting Data to Monod's Law** Table 3.A.1 shows some data for the growth rate of paramonium as a function of the paramonium concentration. Fit the data to Monod's law (Monod [1942]).

$$r_p = \frac{k_1 K_2 [\text{par}]}{1 + K_2 [\text{par}]} \quad (3.A.1)$$

where  $[\text{par}]$  is the paramonium concentration and  $k_1$  and  $K_2$  are constants.

**Solution** There are two methods that people use to solve problems like this:

- Rearranging the equations to get a linear fit and using least-squares
- Doing nonlinear least-squares

I prefer the latter, but I wanted to give a picture of the former.

There are two versions of the linear plots:

- Lineweaver–Burke plots
- Eadie–Hofstee plots

In the Lineweaver–Burke method, one plots  $1/r$  against  $1/\text{concentration}$ . Rearranging equation (3.A.1) shows

$$\frac{1}{r_p} = \frac{1}{k_1 K_2 [\text{par}]} + \frac{1}{k_1} \quad (3.A.2)$$

Therefore, a plot of  $1/r_p$  versus  $1/[\text{par}]$  should be a straight line. The intercept should be  $\frac{1}{k_1}$ . The slope should be  $\frac{1}{k_1 K_2}$ . Once  $k_1$  is determined from the intercept,  $K_2$  can be

Table 3.A.1 The rate of paramonium reproduction as a function of the paramonium concentration

Paramonium concentration, $N/\text{cm}^3$	Rate, $N/\text{cm}^3\text{-hour}$	Paramonium concentration, $N/\text{cm}^3$	Rate, $N/\text{cm}^3\text{-hour}$
2	10.4	16	36
3.6	12.8	16.6	46.4
4	23.2	19	59.2
5.2	17.6	20	62.4
7.8	46.4	23.8	62.4
8	23.2	26	57.6
8	46.4	30.4	108.8
11	32	31	80
14.4	34.4	31.2	61.6
15.6	44.8	31.6	109.6
15.6	63.2	39.2	103.2

Source: Data of Meyers (1927).