18.3 Cvičení 3

Rozhodování o typu extrému (min/max?)

$$I[y] = \int dx\, L(x, y(x), y'(x))$$

analogie s funkcemi

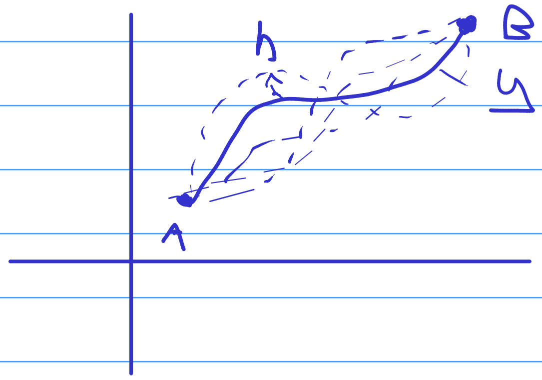
$x \to f(x)$

$$\frac{\partial f}{\partial x} = 0;\ \frac{\partial^2 f}{\partial x^2} > 0$$

funkcionály

$$\frac{\delta I}{\delta y} = 0 \Leftrightarrow \text{EL-rce}$$

$$\frac{\delta^2 I}{\delta y^2} > 0 \ ?$$



$$\delta I[y,h] = \frac{d}{d\varepsilon} I[y+\varepsilon h]\Big|_{\varepsilon=0}$$

$$\delta^2 I[y,h] = \frac{d^2}{d\varepsilon^2} I[y+\varepsilon h]\Big|_{\varepsilon=0}$$

$$\delta I = \frac{d}{d\varepsilon}\int L(x, y+\varepsilon h, y'+\varepsilon h')\,dx$$

$$= \int \frac{\partial L}{\partial y} h + \frac{\partial L}{\partial y'}\cdot h' \, dx$$

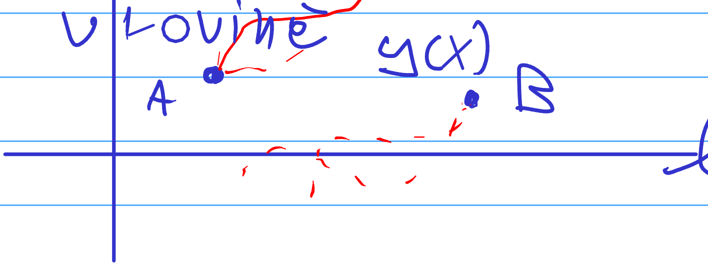
$$\delta^2 I = \int \frac{\partial^2 L}{\partial y^2} h^2 + 2\frac{\partial^2 L}{\partial y\,\partial y'} h\cdot h' + \frac{\partial^2 L}{\partial y'^2} h'^2$$

$$\delta^2 I[h] = \int \left(\frac{\partial^2 L}{\partial y^2} - \cancel{2}\frac{d}{dx}\frac{\partial^2 L}{\partial y\,\partial y'}\right) h^2 + \frac{\partial^2 L}{\partial y'^2} h'^2$$

$$\frac{\partial^2 L}{\partial y\,\partial y'} h \frac{dh}{dx} + \cancel{\frac{\partial^2 L}{\partial y\,\partial y'} h h'}$$

$$-\frac{d}{dx}\left(\frac{\partial^2 L}{\partial y\,\partial y'} h\right) h = -\frac{d}{dx}\left(\frac{\partial^2 L}{\partial y\,\partial y'}\right) h^2 - \cancel{\frac{\partial^2 L}{\partial y\,\partial y'} h' h}$$

Př: nejkratší spojnice 2 bodů v rovině



$$\ell[y] = \int \sqrt{1+(y'(x))^2}\,dx$$

1) $\delta \ell$; EL ; $y_s = ?$

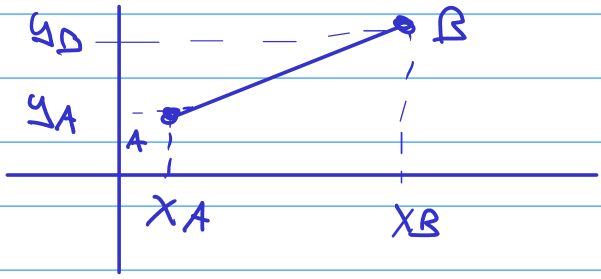
$$L = \sqrt{1+(y'(x))^2}$$

EL $$\frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial y'}$$

$$0 = \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) \qquad \int$$

$$\frac{y'}{\sqrt{1+(y')^2}} = K$$

$$\boxed{y' = \sqrt{\frac{K}{1-K}} = C} \; ; \quad y = Cx + b$$



$$y(x_B) = y_B$$
$$y(x_A) = y_A$$

$$\boxed{y_S = y_A + (x - x_A)\frac{y_A - y_B}{x_A - x_B}}$$

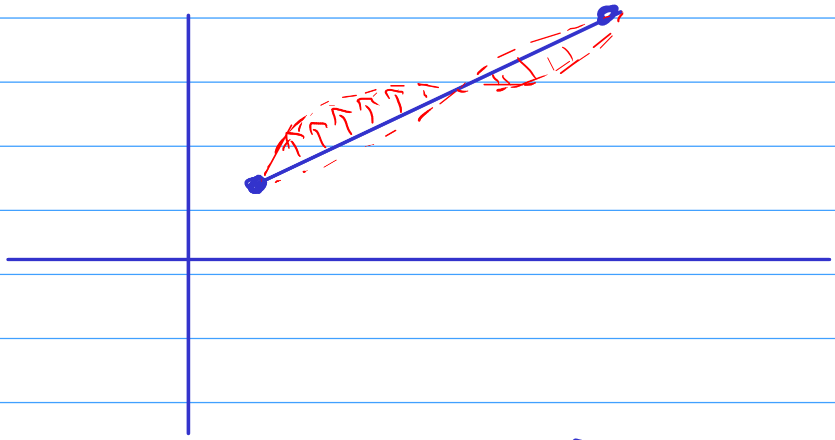
$$\delta^2 I\Big|_{y=y_S} = \int \frac{\partial}{\partial y'}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) h'^2 \, dx$$

$$= \int \left[\frac{1}{\sqrt{1+(y')^2}} + y' \cdot \left(-\frac{y'}{(1+(y')^2)^{\frac{3}{2}}}\right)\right] h'^2 \, dx$$

$$= \int \frac{1+\cancel{y'^2} - \cancel{y'^2}}{\left(1+(y')^2\right)^{\frac{3}{2}}} \, h'^2 \, dx$$

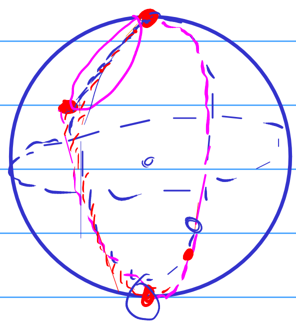
$$\delta^2 I \Big|_{y=y_s} = \int \frac{1}{\left(1+(y')^2\right)^{\frac{3}{2}}} \, h'^2 \, dx$$

$$= \int \frac{1}{(1+c^2)^{\frac{3}{2}}} \, h'^2 \, dx > 0$$



krivky v rovine $\quad L = \sqrt{dx^2+dy^2}$

spec. teor rel $\quad L = \sqrt{dx^2+dy^2+dz^2-dt^2}$
(t, x, y, z)



2 Př: Harmonický oscilátor

$$I[y] = \int dx \left((y')^2 - \omega^2 y^2 \right)$$

$\omega^2 = \frac{k}{m}$

$x(t)$

$y(x)$

$\downarrow$ EL

$$y'' + \omega^2 y = 0$$

$$y_S = A\cos\omega x + B\sin\omega x$$

$$\delta^2 I = \int h^2(-2\omega^2) + h'^2\, 2 \quad dx$$

$$\frac{1}{2}\delta^2 I[h]\Big|_{y=y_S} = \int h'^2 - \omega^2 h^2 \quad dx$$

I $\frac{\partial^2 L}{\partial y'^2} = 1 > 0$ ✓

II h řeší EL $\quad h'' + \omega^2 h = 0$

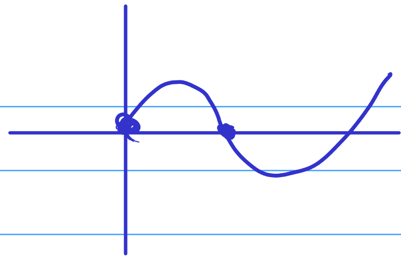
$h = A\sin\omega x + B\cos\omega x$

$x_A, x_B \qquad h(x_A) = h(\tilde{x}_A) = 0$

$\tilde{x}_A$

$$h = \frac{\sin\omega(x - x_A)}{\omega}$$

$\tilde{x}_A$ tak $h(\tilde{x}_A) = 0$?

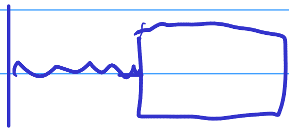


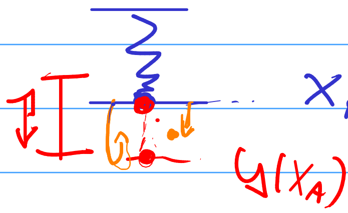
$$\sin\omega(x-x_A)=0$$

$$\omega(x-x_A)=\pi$$

$$\omega x-\omega x_A=\pi$$

$$x=\frac{\pi+\omega x_A}{\omega}$$





$x_A=0$

$\tilde{x}_A=\frac{\pi}{\omega}$

$$y = A\sin\omega x + B\cos\omega x$$