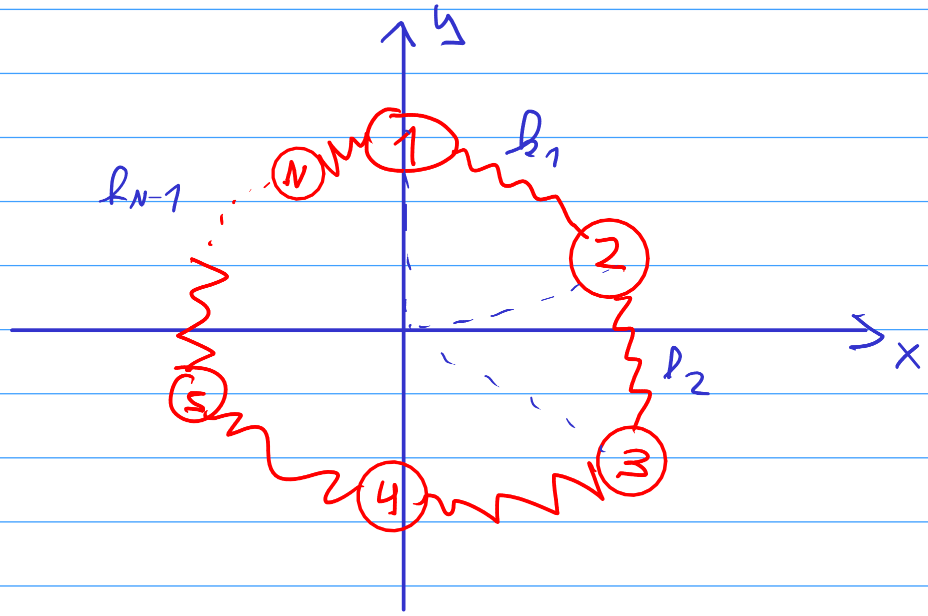
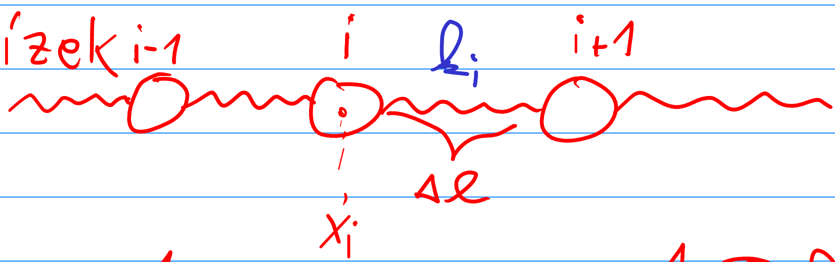
4 cvičení soustava N harmonických oscilátorů



1D řetízek



$$T = \frac{1}{2}\sum m_i \dot{x}_i^2 \; ; \; V = \frac{1}{2}\sum k_i \underbrace{(x_{i+1} - x_i)}_{\Delta l}^2$$

Souřadnice

① x_1, y_1 ; $U_1 = (x_1, y_1)$

② x_2, y_2 ; $U_2 = (x_2, y_2)$

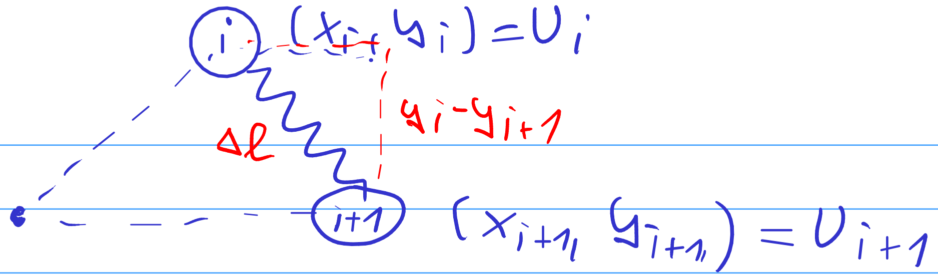
⋮

Ⓝ x_N, y_N ; $U_N = (x_N, y_N)$

$$\vec{U} = (U_1, \dots, U_N) \, , \quad U_i$$

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i (\dot{x}_i^2 + \dot{y}_i^2) = \frac{1}{2}\sum m_i \dot{U}_i^2$$

$$V = \frac{1}{2}\sum k_i (\Delta l_i)^2$$



$$\Delta \ell = \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2}$$
$$= \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$$
$$= \sqrt{(U_{i+1} - U_i)^2}$$

$$(x_{i+1}, y_{i+1}) - (x_i, y_i)$$

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i (\dot{x}_i^2 + \dot{y}_i^2) = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{U}_i^2$$

$$V = \frac{1}{2} \sum_{i=1}^{N-1} k_i (U_{i+1} - U_i)^2 + \frac{1}{2} k_N (U_N - U_1)^2$$

$$\rightarrow (U_{i+1})^2 - 2 U_{i+1} U_i + U_i^2$$

$$\vec{U} = (U_1, \dots, U_N)$$

$$T = \frac{1}{2} \dot{\vec{U}} M \dot{\vec{U}}^T$$

$$= \frac{1}{2} (\dot{U}_1 \quad \dots \quad \dot{U}_N) \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_N \end{pmatrix} \begin{pmatrix} \dot{U}_1 \\ \vdots \\ \dot{U}_N \end{pmatrix}$$

$$V = \frac{1}{2} \vec{U} K \vec{U}^T$$

$$\frac{1}{2} (U_1 \dots U_N) \begin{pmatrix} k_1 + k_N & -k_1 & \dots & -k_N \\ -k_1 & k_2 + k_3 & \ddots & \\ & \ddots & \ddots & \\ -k_N & & & k_{N-1} + k_N \end{pmatrix} \begin{pmatrix} U_1 \\ \vdots \\ U_N \end{pmatrix}$$

$$k_i = k \quad \begin{pmatrix} 2k & -k & & & -k \\ -k & 2k & -k & & \\ & -k & 2k & \ddots & \\ & & \ddots & \ddots & -k \\ -k & & & -k & 2k \end{pmatrix}$$

$$= k \underbrace{\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ -1 & & & -1 & 2 \end{pmatrix}}_{A}$$

$$L = T - V = \frac{1}{2} \dot{u}_i M^i{}_j \dot{u}^j - \frac{1}{2} u_i K^i{}_j u^j$$

$$\frac{\partial L}{\partial u_k} = \frac{d}{dt} \frac{\partial L}{\partial \dot{u}_k}$$

$$\frac{\partial L}{\partial u_k} = -\frac{1}{2} \delta_{ik} K^i{}_j u^j - \frac{1}{2} u_i K^i{}_j \delta^j_k$$

$$= -\frac{1}{2} \underline{K^k{}_j} u^j - \frac{1}{2} u_i \underline{K^i{}_k} = -\vec{u} K$$

$$= u^i K_{ik}$$

$$K = K^T \iff K_{ij} = K_{ji}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{u}_k} = \ddot{u} M$$

$$M\ddot{U} = -KU$$

$m_i = m$
$k_i = k$

$$\ddot{U}\, m\, \mathbb{1} = -k \begin{pmatrix} -1 & 2 & -1 \\ & -1 & \ddots \\ & & -1 & \ddots \end{pmatrix} U \,/\, m$$

$$\omega^2 = \frac{k}{m}$$

$$\ddot{\vec{U}} = -\omega^2 A \vec{U}$$

$A \to$ vl. vektory $\qquad V = c\, e^{-\omega^2 A t}$
vl. hodnoty

trik z QM.

S ; $AS = SA$ $\quad \langle v_1, \ldots, v_N \rangle$

vl. hodnoty $S \to$ vl. vektory $S =$
vl. vektory $A \to$ vl. hodnoty A

$$A v_A = \lambda_A v_A$$

$$S \tilde{v}_B = \lambda_B \tilde{v}_B$$

$$AS = SA$$

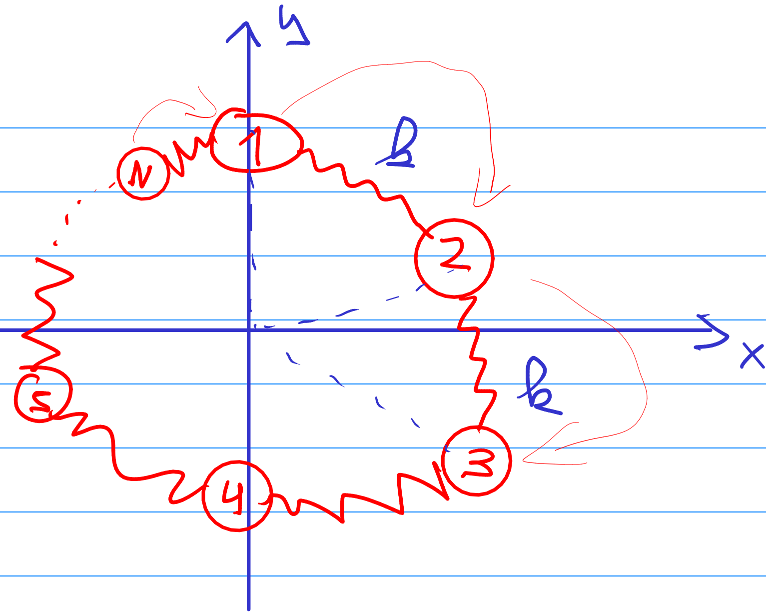
$$A S v_A = S A v_A$$

$$A(S v_A) = \lambda_A (S v_A)$$

Cvičení 5

$\vec{U} = (U_1, \dots, U_N)$

$\tilde{U} = (U_N, U_1, U_2, \dots, U_{N-1})$



$$US = \tilde{U}$$

$$(U_1 \dots U_N)\begin{pmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ 1 & & & & 0 \end{pmatrix} = (U_N\, U_1 \dots U_{N-1})$$

$$-\omega^2 A U = \ddot{U} \qquad \Big/ \cdot S$$

$$-\omega^2 SAU = S\ddot{U}$$

$$-\omega^2 A\tilde{U} = \ddot{\tilde{U}}$$

$$\boxed{SA = AS}$$

$$\underbrace{\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ 1 & & & \end{pmatrix}}_{S}\underbrace{\begin{pmatrix} 2 & -1 & & -1 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ -1 & & -1 & 2 \end{pmatrix}}_{A} = \begin{pmatrix} -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots \\ & & & & \\ & & & -1\;2 & -1 \\ -1 & & & & -1\;2 \\ 2 & -1 & & & -1 \end{pmatrix}$$

$AS = SA$

$Au_\lambda = \lambda^A u_\lambda$ $\qquad A S u = S A u$

$Sv = \lambda^S v$ $\qquad = S \lambda^A u$

$\qquad = \lambda^A S u$

u je vl. v. A s λ^A

Su je vl. v. A s λ^A

je-li λ^A 1 násobná $\langle u \rangle$ ~ 1 dim

$\boxed{Su = c\,u}$

$\Rightarrow$ vl. hodnoty matice $S = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & & & 0 \end{pmatrix}$

$S^N = 1$

$Su = \lambda u$ $\quad ; \quad S^{N-1}$

$S^N u = \lambda^N u$

$u = \lambda^N u \Rightarrow \lambda^N = 1$

$\det(S - \lambda 1) = 0$

$$\begin{vmatrix} -\lambda & 1 & & & \\ & -\lambda & \ddots & & \\ & & -\lambda & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & -\lambda \end{vmatrix} = -\lambda \cdot (-1)^{1+1} \cdot D_{N-1}$$

$$+ 1 \cdot (-1)^{N+1} \cdot S_{N-1} \qquad (*)$$

$$D_{N-1} = \begin{vmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & -\lambda \end{vmatrix} = -\lambda \cdot D_{N-2}$$

$$= -\lambda^{N-3} \cdot D_2 = -\lambda^{N-1}$$

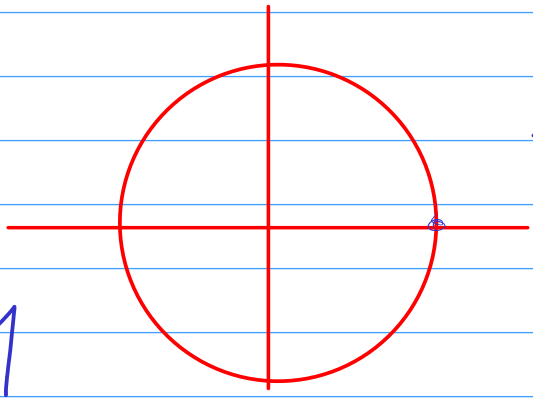
$$D_2 = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = -\lambda^2$$

$$S_{N-1} = \begin{vmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{vmatrix} = 1\, S_{N-2} = S_2 = 1$$

$$\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$(*) = (-\lambda)^N + (-1)^{N-1} = (-1)^N \underline{\underline{(\lambda^N - 1)}} = 0$$

$$\lambda^N = 1$$



$$\underline{\lambda_k = e^{i2\pi \frac{k}{N}}}$$

$$\lambda = r e^{i\varphi}$$

$$r e^{iN\varphi} = 1$$

$$e^{iN\varphi} = e^{i2\pi k}$$

$$S \vec{u}_k = \lambda_k u_k$$

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & \\ 1 & & 0 & 1 \end{pmatrix} \vec{u}_k = e^{2i\pi \frac{k}{N}} \quad u_k \qquad k \in \{1, N\}$$

$\vec{U} = (U_1, \ldots, U_N)$

$\vec{U}_k = (U_1^k, \ldots, U_N^k)$

$$U_1^k = e^{2\pi i \frac{k}{N}} U_N^k$$

$$U_2^k = e^{2\pi i \frac{k}{N}} U_1^k$$

$$U_3^k = \text{---}\,||\,\text{---}\; U_2^k$$

$$\vdots$$

$$\left(e^{\frac{2\pi i k}{N}}\right)^0 \quad U_N^k = \text{---}\,||\,\text{---}\; U_{N-1}^k$$

$$\left(1, \left(e^{2\pi i \frac{k}{N}}\right)^1, \left(e^{2\pi i \frac{k}{N}}\right)^2, \left(e^{2\pi i \frac{k}{N}}\right)^{N-1}\right)$$

$= U_k$

$$U^1 = \left(1, e^{2\pi i \frac{1}{N}}, \left(e^{2\pi i \frac{1}{N}}\right)^2, \ldots, \right)$$

$$U^2 = \left(1, e^{2\pi i \frac{2}{N}}, \ldots \right)$$

$\vdots$

Vl vektory $S \Rightarrow$ Vl. vektory A

$$A \cdot \vec{U}_k = \lambda_k \vec{U}_k$$

$$A \begin{pmatrix} U_1^k \\ \vdots \\ U_N^k \end{pmatrix} = \lambda_k \begin{pmatrix} U_1^k \\ \vdots \\ U_N^k \end{pmatrix}$$

$$2U_1^k - U_2^k - U_N^k = \lambda_k U_1^k$$

$$2 \cdot 1 - 1 \cdot e^{2\pi i \frac{k}{N}} - e^{\frac{2\pi i k}{N}(N-1)} = \lambda_k 1$$

$$2 - e^{2\pi i \frac{k}{N}} - e^{2\pi i k - \frac{2\pi i k}{N}} = \lambda_k$$

$$2 - \underbrace{e^{2\pi i \frac{k}{N}}}_{\cos(\,)+i\sin(\,)} - \underbrace{e^{-\frac{2\pi i k}{N}}}_{\cos(\,)-i\sin(\,)} = \lambda_k$$

$$2 - \left(2\cos\frac{2\pi k}{N}\right) = \lambda_k$$

$$2\left(1 - \cos\frac{2\pi k}{N}\right) = \lambda_k \quad \in \mathbb{R} \quad \checkmark$$

Nepovinný DÚ $\qquad t, x$

$$L = \frac{1}{2} m x \ddot{x} + \frac{1}{2} k x^2$$

přičtěte k L nějaký triviální L_0 tak aby výsledný $L + L_0 \not\ni \ddot{x}$ ale jenom $x, \dot{x}$