6 cvičení Spec. typy Lagrangiánů

1) triviální Lagrangiány

$$L = \frac{1}{2} m x \ddot{x}$$

$L_0 = ?$ tak že $L + L_0 \sim (t, x, \dot{x})$

$$L_0 = \frac{d}{dt}\left(f(t, x, \dot{x})\right)$$

$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial \dot{x}}\ddot{x}$$

$$-\frac{1}{2} m \dot{x} x$$

$$\int \frac{1}{2} m \dot{x}^2 \, dt = \frac{1}{2} m \int \dot{x} \dot{x} \, dt$$

$$= -\frac{1}{2} m \int x \ddot{x}$$

- per partes
- inverzní variační problém

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial y^2}{\partial x^2} = 0$$

$y = y(x, t)$

$$L = \frac{1}{2} y \left(\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} \right) + \frac{d}{dt} f(y, y_x, y_t)$$

$$L = \frac{1}{2} \left(\left(\frac{\partial y}{\partial t}\right)^2 - \left(\frac{\partial y}{\partial x}\right)^2 \right)$$

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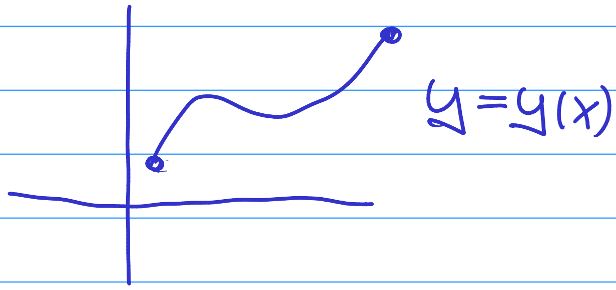
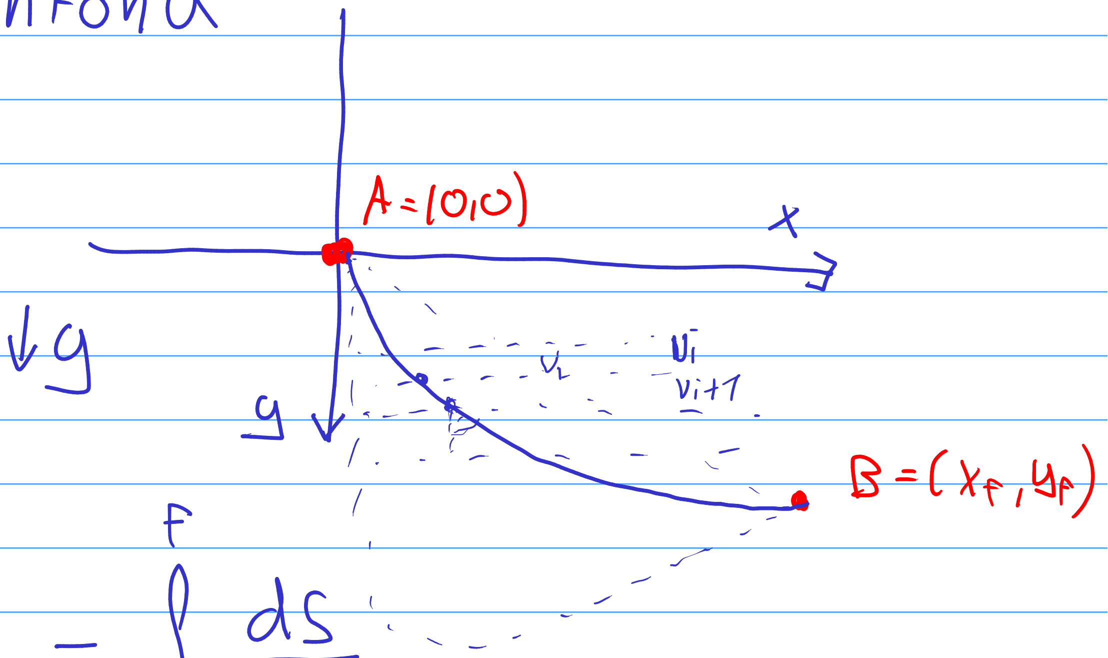
2) $L = L(y, y')$



Brachistochrona



$$T = \int_0^T dt = \int_0^f \frac{ds}{v}$$

$$\frac{1}{2} m v^2 = m g y$$

$$v = \sqrt{2gy}$$

$$I[y] = \int_0^{x_f} \underbrace{\frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gy}}}_{L} dx$$

$$L = T - V \qquad L = \frac{\sqrt{1+y'(x)^2}}{\sqrt{2gy}}$$

$$\frac{\partial L}{\partial y'} y' - L = \text{konst}$$

$$\frac{1}{\sqrt{2gy}} \frac{y'}{\sqrt{1+y'^2}} \cdot y' - \frac{\sqrt{1+y'(x)^2}}{\sqrt{2gy}} = k$$

$$\Rightarrow \frac{y'^2 - (1+y'^2)}{\sqrt{2gy}\sqrt{1+y'^2}} = \frac{-1}{\sqrt{2gy(1+y'^2)}} = k$$

$$2gy(1+y'^2) = C$$

$$\vdots$$

$$y' = \sqrt{\frac{K-y}{y}} \qquad \cos^2 x + \sin^2 x = 1$$

$$X = X_0 + \int \sqrt{\frac{y}{K-y}}\, dy = \left| \begin{array}{l} y = K\sin^2 t \\ dy = 2K\sin t \cos t\, dt \end{array} \right|$$

$$\sqrt{\frac{y}{K-y}} = \sqrt{\frac{y}{K(1-\frac{y}{K})}}$$

$$\Rightarrow X = X_0 + \int dt \sqrt{\frac{K\sin^2 t}{K\cancel{\cos^2 t}}}\, K\sin t \cancel{\cos t}$$

$$x = x_0 + \int dt \, K \sin^2 t$$

$$x = x_0 + Kt - \frac{K}{2}\sin 2t \qquad \left(\begin{array}{l} x(0)=0 \\ \Rightarrow x_0 = 0 \end{array} \right)$$

$$y = K\sin^2 t = \frac{k}{2} - \frac{K}{2}\cos 2t$$

$$x = R(\theta - \sin\theta)$$

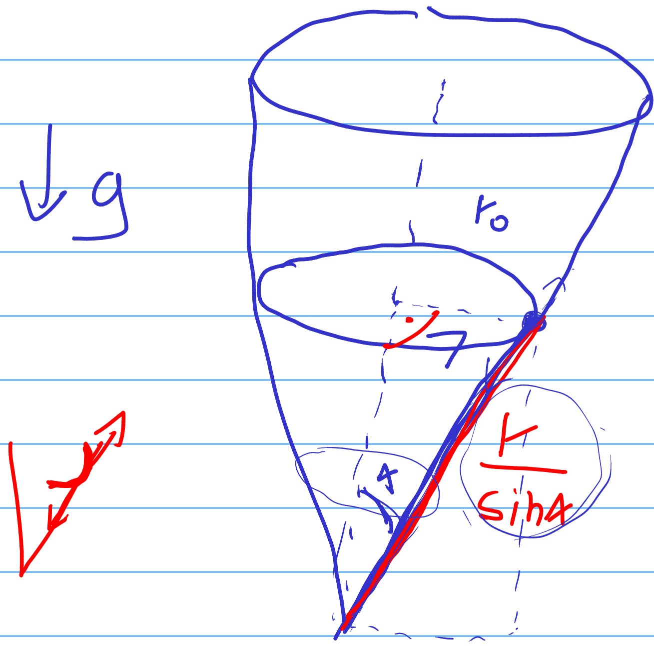
$$y = R(1 - \cos\theta)$$

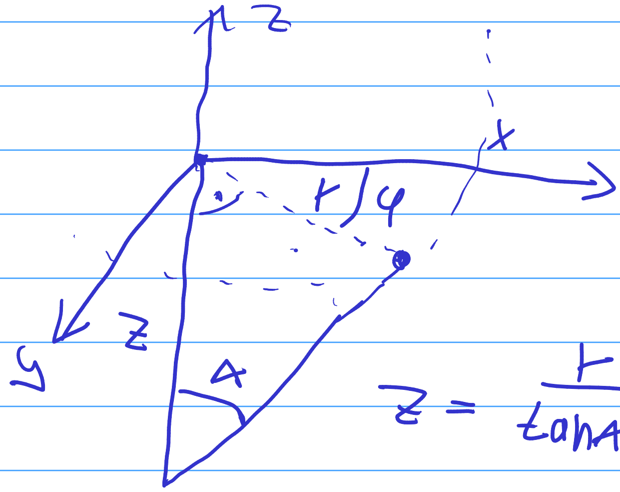
$$\left. \begin{array}{l} x(T) = x_f \\ y(T) = y_f \end{array} \right\} \quad x = \frac{x_f t}{T - \frac{\sin T}{2}} - \frac{x_f \sin 2t}{2T - \sin 2T}$$

$$x_p = \frac{x_f}{T - \frac{\sin 2T}{2}} \left(t - 2\sin 2t \right)$$

$$y_p = \frac{y_f}{2\sin^2 T} - \frac{1}{2} y_f \frac{\cos 2t}{\sin^2 T}$$

3) $L = L(x, y')$ $\quad L \sim y$





$$z = \frac{r}{\tan A}$$

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$z = \frac{r}{\tan A}$$

$$L = \frac{1}{2} m v^2 - m g z$$

$$L = \frac{1}{2} m\left((\dot r\cos\varphi - r\sin\varphi\,\dot\varphi)^2 + (\dot r\sin\varphi + r\cos\varphi\,\dot\varphi)^2 + \frac{\dot r^2}{\tan^2 A}\right) - mg\frac{r}{\tan A}$$

$$\therefore\ L = \frac{1}{2} m\left(\dot r^2 + r^2\dot\varphi^2 + \frac{\dot r^2}{\tan^2 A}\right) - mg\frac{r}{\tan A}$$

$$L = \frac{1}{2} m\left(\frac{\dot r^2}{\sin^2 A} + r^2\dot\varphi^2\right) - mg\frac{r}{\tan A}$$

$$1 + \frac{1}{\tan^2 A} = \frac{\tan^2 A + 1}{\tan^2 A} = \frac{\frac{\sin^2 A}{\cancel{\cos^2 A}} + \frac{\cos^2 A}{\cancel{\cos^2 A}}}{\frac{\sin^2 A}{\cancel{\cos^2 A}}}$$

$L = L(x, y, y')$

$(x, y) \to (u, v)$

$$L = \frac{y^2}{x^2 y'^2}$$

$y \to f(u, v)$

$x \to f(u, v)$

$$\frac{\partial L}{\partial v} = 0$$

zz

$L \not\ni \varphi$

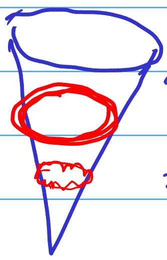
$$\frac{\partial L}{\partial \dot\varphi} = konst$$

$$\boxed{mr^2\dot\varphi = konst = L}$$

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right)$$

$$-\frac{\cancel{m}g}{\tan A} + \cancel{m}r\dot\varphi^2 = \frac{\cancel{m}}{\sin^2 A}\ddot r$$

$$\ddot r = \frac{L^2 \sin^2 A}{m^2 r^3} - g\cos A \sin A \quad \Big| \cdot \dot r$$



$r = r(t)$

1) $r = r_0 = konst$ ω

2) $r = r_0 + \varepsilon\delta(t)$ Ω

$\Big\}$ Harard 6.18

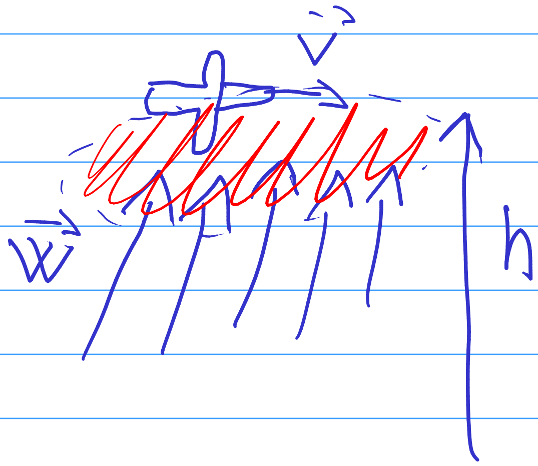
$$\ddot r\dot r = \dot r A \frac{1}{r^3} - B\dot r$$

$$\frac{d}{dt}\left(\frac{1}{2} \dot r^2\right) = -\frac{A}{2}\frac{d}{dt}\left(\frac{1}{r^2}\right) - B\frac{d}{dt}(r) \quad \Big| \int$$

$$\frac{1}{2}\dot r^2 + \frac{A}{2r^2} + Br = C$$

$$\frac{dr}{dt} = \sqrt{-\frac{A}{r^2} - 2Br + C}$$

Zadaní na příště



zadaný čas T jaká bude trajekto
rie letadla tak aby S (uzavřená trajekt)
byla co největší