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Analogies Between the Chaplygin Problem and the Kepler Problem

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Chaplygin's problem of a powered airplane in a steady cross wind, and Kepler's problem of a satellite subject to a central attraction force, both lead to elliptic orbits. The resulting analogies are discussed, established, and tabulated. Then Kepler's problem is reformulated using Chaplygin's approach, and finally Chaplygin's problem is reformulated as a central force problem.

1 Introduction

Chaplygin's is a classical problem of the calculus of variations, involving the kinematics of an aircraft in a steady cross wind, and formulated as an isoperimetric problem in parametric form. Kepler's is a classical problem of dynamics, involving the kinetics of a satellite in an inverse square central force field. It is typically posed as a vectorial problem involving force and acceleration. Both have as result an elliptical orbit, inviting the obvious question as to how far an analogy can be pursued. Specific questions that might be posed include: Is Chaplygin's problem, by any chance, also a central force problem? Can Kepler's problem be posed as a variational problem similar to Chaplygin's? Since both are problems in Newtonian mechanics, both can obviously be treated via Hamilton's principle. What then is the potential energy for Chaplygin's problem? What is the variational statement for Keplers' problem using an isoperimetric approach?

2 Chaplygin's Problem

Chaplygin's problem involves an airplane flying in a horizontal plane, with an air velocity v_a (relative to air) of constant magnitude (Grigorian, 1965). The airplane is required to follow a closed path over a given period τ. What path should it follow if the area boundary by the path is to be maximum, given that a constant wind velocity $v_w < v_a$ is prevailing?

Let y be the direction of the wind velocity and suppose that α is the angle between the direction of the axis of the airplane and x. Let the closed path be described parametrically by

$$x = x(t) \qquad y = y(t)$$

If we associate t with time then the components of the ground velocity v (with respect to earth) of the airplane become (Figure 1)

$$\dot{x} = v_a \cos\alpha \tag{1a}$$

$$\dot{y} = v_a \sin\alpha + v_w \tag{1b}$$

The area bounded by the airplane path is given by

$$A = \frac{1}{2}\int_0^\tau (x\dot{y} - \dot{y}x)dt \tag{2}$$

We require to maximize the area A subject to the nonholonomic constraint equations (1). Evidently to locate the airplane we need to know x, y and α. Hence we allow variations of α as well as x and y. Thus we construct a modified area functional (Pars, 1962) and express its variation as follows:

$$\delta\int\left[\frac{1}{2}(x\dot{y}-y\dot{x}) + \lambda_1(\dot{x}-v_a\cos\alpha) + \lambda_2(\dot{y}-v_a\sin\alpha-v_w)\right]dt = \delta\int F\,dt = 0 \tag{3}$$

in which $\lambda_1(t)$ and $\lambda_2(t)$ are Lagrange multipliers. The Euler-Lagrange equations of the functional $\int F\,dt$ emerges as follows:

For δx:
$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = \frac{1}{2}\dot{y} - \frac{d}{dt}\left(-\frac{1}{2}y+\lambda_1\right) = 0 \tag{4a}$$

δy:
$$\frac{\partial F}{\partial y} - \frac{d}{dt}\frac{\partial F}{\partial \dot{y}} = -\frac{1}{2}\dot{x} - \frac{d}{dt}\left(\frac{1}{2}x+\lambda_2\right) = 0 \tag{4b}$$

$\delta\alpha$:
$$\frac{\partial F}{\partial \alpha} = (\lambda_1\sin\alpha - \lambda_2\cos\alpha)v_a = 0 \tag{4c}$$

$\delta\lambda_1$:
$$\frac{\partial F}{\partial \lambda_1} = \dot{x} - v_a\cos\alpha = 0 \tag{4d}$$

$\delta\lambda_2$:
$$\frac{\partial F}{\partial \lambda_2} = \dot{y} - v_a\sin\alpha - v_w = 0 \tag{4e}$$

From the first two equations we have, on integration,

$$\lambda_1 = y + c_1 \qquad \lambda_2 = -(x+c_2) \tag{5}$$

where c_1 and c_2 are constants. Substituting into equation (4c)

$$(x+c_2)\cos\alpha + (y+c_1)\sin\alpha = 0 \tag{6}$$

To satisfy this equation we let

$$x + c_2 = r\sin\alpha \qquad y + c_1 = -r\cos\alpha \tag{7}$$

and then from the constraint equation (1)

$$\begin{aligned} \dot{r}\sin\alpha + r\dot{\alpha}\cos\alpha &= v_a\cos\alpha \\ -\dot{r}\cos\alpha + r\dot{\alpha}\sin\alpha &= v_a\sin\alpha + v_w \end{aligned} \tag{8}$$

Multiplying the first by $\sin\alpha$ and the second by $\cos\alpha$, and subtracting, we obtain

$$\dot{r} = -v_w\cos\alpha \tag{9}$$

But from the first of equations (1)

$$\cos\alpha = \frac{1}{v_a}\dot{x} \tag{10}$$

hence

$$\dot{r} = -\frac{v_w}{v_a}\dot{x} \tag{11}$$

Integrating

$$r = -\frac{v_w}{v_a}x + p \tag{12}$$

where p is an arbitrary constant. The x-component of the airplane's position, using polar coordinates emanating from a focus O (Figure 1), is $x = r\cos\theta$, such that equation (12) can be rearranged to read

$$r = \frac{p}{1+\frac{v_w}{v_a}}\cos\theta \tag{12}$$

which is the polar equation of conic (an ellipse if $v_w < v_a$) of eccentricity

$$\varepsilon = \frac{v_w}{v_a} \tag{13}$$

of semi-parameter (semi-latus-rectum)

$$p \tag{14}$$

of semi-major axis

$$a = \frac{p}{1-\frac{v_w^2}{v_a^2}} \tag{15}$$

and of semi-minor axis

$$b = \frac{p}{\left(1-\frac{v_w^2}{v_a^2}\right)^{1/2}} \tag{16}$$

The area A of the ellipse covered is obtained from

$$A = \pi ab = \pi\frac{p^2}{\left(1-\frac{v_w^2}{v_a^2}\right)^{3/2}} \tag{17}$$

The time period to circumnavigate the area A can be shown to be

$$\tau = 2\pi\frac{p}{v_a\left(1-\frac{v_w^2}{v_a^2}\right)^{3/2}} \tag{18}$$

For an eccentricity of $\varepsilon = 0$ (i.e. for vanishing wind velocity v_w) the ellipse degenerates into a circle. We conclude that the greater the wind speed v_w, the flatter the ellipse (Figure 1).

From equations (1), the Cartesian acceleration components are

$$\ddot{x} = -v_a\dot{\alpha}\sin\alpha$$
$$\ddot{y} = v_a\dot{\alpha}\cos\alpha \qquad (19)$$

which when combined, represent the airplane's acceleration

$$\sqrt{\ddot{x}^2+\ddot{y}^2} = v_a\dot{\alpha} = \frac{v_a^2 p}{r^2} \qquad (20)$$

Since v_w = constant in direction and magnitude, and v_a = constant in magnitude only, it can be concluded readily that the airplane's acceleration is the result of the change of direction of the v_a vector. This change, which is perpendicular to the v_a vector, always points towards the origin 0 of the coordinate system. As a consequence of which we find

$$\alpha = \theta + 90° \qquad (21)$$

and conclude that the airplane's acceleration is pointing towards the origin O of the coordinate system (Figure 1). Consequently the airplane's motion is a "central acceleration" type motion, analogous the "central force" type motion of e.g. planets.

The choice of polar coordinates emanating from the focus 0 of the ellipse, also means that the integration constants in equations (6) vanish, i.e.

$$c_1 = c_2 = 0 \qquad (24)$$

Dynamics

It is important to note that Chaplygin's problem is purely *kinematic*. All equations, and the solutions obtained, involve only kinematic statements. The Euler-Lagrange equations (4) are all in terms of velocities. In view of our intended comparison with the Kepler problem, it is of interest to have a look at the *dynamics* of the Chaplygin's problem. In particular we are interested in the *forces* which are involved in producing the kinematics of Chaplygin's airplane. Using the polar coordinate angle θ, equation (21), rather than the air speed angle α, and using equations (4d, e), the derivatives of equations (6) become

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta} = -v_a\sin\theta \qquad (23a)$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta = v_a\cos\theta + v_w \qquad (23b)$$

Differentiating again

$$\ddot{x} = \left(\ddot{r}-r\dot{\theta}^2\right)\cos\theta - \left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)\sin\theta = -v_a\dot{\theta}\cos\theta \qquad (24a)$$

$$\ddot{y} = \left(\ddot{r}-r\dot{\theta}^2\right)\sin\theta + \left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)\cos\theta = -v_a\dot{\theta}\sin\theta \qquad (24b)$$

Multiplying one equation with $\cos\theta$ and the other with $\sin\theta$, and then doing the reverse, adding, and multiplying with the airplane mass m the equations of the motion in polar coordinates are obtained

$$m\left(\ddot{r}-r\dot{\theta}^2\right) = -mv_a\dot{\theta} \qquad (25a)$$

$$m\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right) = 0 \qquad (25b)$$

It becomes immediately obvious that the airplane is acted upon by a resultant force which is directed towards the centre O of the coordinate system

$$F_r = -mv_a\dot{\theta} \tag{26}$$

or, in other words Chaplygin's problem can be looked upon as a *central force* problem. In order to gain further information, we rewrite equation (25b) into

$$2\frac{\dot{r}}{r} = -\frac{\ddot{\theta}}{\dot{\theta}} \tag{27a}$$

which, upon integration, gives

$$r^2\dot{\theta} = h = \text{constant} \tag{27b}$$

where the integration constant h represents the airplane's specific angular momentum about the centre 0 of attraction, corroborating the well-known fact that central force motion is characterized by angular momentum constancy. Now , with the help of equations (27), the central force (26) can be written

$$F_r = -\frac{mv_a h}{r^2} \tag{28}$$

Since m, v_a, and h are all constants it can be seen that the airplane moves indeed in an *inverse square force field.*

It is interesting to note that, in principle, the airplane's pilot needs only follow a prescribed elliptic path at a constant height and a constant air speed. The obviously very complicated force system (thrust, drag, weight, lift, rudder and aileron effects etc.) acting on the plane, will then adjust itself such, that the resultant is a pure central force, as given by equation (28). With the help of equations (27), equations (25) can be written

$$\ddot{r} - r\dot{\theta}^2 = -\frac{v_a h}{r^2} \tag{29a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{29b}$$

providing us with a relationship involving accelerations.

3 Kepler's Problem

Variational Formulation

Kepler's problem of the motion of a satellite in an inverse square central force field, is usually solved without recourse to variational methods (Rimrott, 1989). For purposes of the present study let us state Kepler's problem in variational form, for which Hamilton's principle (Goldstein, 1980)

$$\delta\int Ldt = 0 \tag{30}$$

is the obvious choice, with the Lagrangian

$$L = T - V \tag{31}$$

and a kinetic energy of

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\theta\right) \tag{32}$$

and a potential energy of

$$V = -\frac{\mu m}{r} \tag{33}$$

where μ is the gravitational attraction constant, with $\mu = 398601 \frac{\text{km}^3}{\text{s}^2}$ for the earth as attractor. With equations (31), (32) and (33) we can now express the satellite system's variation (30) as follows:

$$\delta \int L dt = \delta \int m\left[\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{\mu}{r}\right] dt = 0 \tag{34}$$

When one compares equations (34) and (3), then one notices, apart from polar coordinates in the one and Cartesian coordinates in the other, that the functional for Kepler's problem is in action units, i.e. Js, while the functional (3) for Chaplygin's problem was in area units, i.e. m^2. This is confirmed when the Euler-Lagrange equations for variation (34) are formed, which emerge as follows:

For δr:
$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\left(r\dot{\theta}^2 - \ddot{r} - \frac{\mu}{r^2}\right) = 0 \tag{35a}$$

For $\delta\theta$:
$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -m\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \tag{35b}$$

They are relations between forces. With the mass cancelled out and rearranged, they result in the well-known differential equations of satellite motion

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} \tag{36a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{36b}$$

Note that the variational approach for Kepler's problem has led to equations (36) involving *accelerations*, while the variational treatment of Chaplygin's problem has led to equations (4) involving *velocities*.

Variational Formulation at Velocity Level

In order to get into a position to treat the Kepler problem in the same fashion as the Chaplygin problem, the former must be raised from the acceleration level to the velocity level.

Beginning with the acceleration equations (36), we integrate equation (36b) to obtain

$$r^2\dot{\theta} = h = \text{constant} \tag{37}$$

and combine equations (37) and (36a) to get

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{h}\dot{\theta} \tag{38}$$

Now realizing that the left hand side of equation (38) represents the radial acceleration component a_r, it can be resolved into Cartesian components

$$\ddot{x} = a_r \cos\theta \tag{39a}$$

$$\ddot{y} = a_r \sin\theta \tag{39b}$$

or with the help of equation (38)

$$\ddot{x} = -\frac{\mu}{h}\dot{\theta}\cos\theta \tag{40a}$$

$$\ddot{y} = -\frac{\mu}{h}\dot{\theta}\sin\theta \tag{40b}$$

Equations (40) are readily integrated and give

$$\dot{x} = -\frac{\mu}{h}\sin\theta + c_3 \tag{41a}$$

$$\dot{y} = \frac{\mu}{h}\cos\theta + c_4 \tag{41b}$$

By letting $c_3 = 0$ we select the coordinate system such that $\frac{dx}{dt} = 0$ at $\theta = 0$ (Figures 1 and 2). Now Kepler's problem has been raised to the level of velocities (41) and is in a form exactly analogous to Chaplygin's problem (1). We can start at equations (1) and proceed. The ratio $\frac{\mu}{h}$ is analogous to v_a, and c_4 is analogous to v_w. Based on this latter analogy, it is not difficult to show that

$$c_4 = \varepsilon\frac{\mu}{h} \tag{41}$$

In choosing this approach, the Kepler problem will then have to be stated in a fashion analogous to the Chaplygin problem, i.e. *Kepler's problem involves a satellite, flying in an orbital plane and subjected to a central force. The satellite is required to complete a closed path during a given period* τ. *What path of a prescribed eccentricity* ε *should it follow if the area A bounded by the path is to be a maximum, given that the central force obeys the inverse square law?*

4 Chaplygin's Problem at Acceleration Level

It must, of course, also be possible to formulate Chaplygin's problem via Hamilton's principle, which means, in effect, that we must find the problem's Lagrangian. With a kinetic energy

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{42}$$

and a potential energy from Table 1

$$V = -\frac{mv_a h}{r} \tag{43}$$

and $L = T - V$, we have

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{mv_a h}{r} \tag{44}$$

Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \tag{45a}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \tag{45b}$$

result in

$$\ddot{r} - r\dot{\theta}^2 = -\frac{v_a h}{r} \quad (46a)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad (46b)$$

i.e. in the set of equations (29) we found previously by other means. It is obviously difficult to perceive of a "potential energy" in case of Chaplygin's problem, and it is only by forcing the problem to be treated as an acceleration problem, that a "potential energy" arises, and that it must have the form given by equation (43).

It is perhaps interesting to note, that the two Lagrangian multipliers of equation (3) are not needed in the acceleration formulation.

5 Conclusions

The analogies resulting from the identical results of the Chaplygin problem and the Kepler problem have been used to cast the former into Kepler formalism, and the latter into Chaplygin formalism, with the result that Chaplygin's problem can be looked upon as an inverse square central force problem, and that Kepler's problem can be presented as an isoperimetric variational problem.

Literature

1. Goldstein, H.: Classical Mechanics, Addison-Wesley, London, (1980), 2nd ed.

2. Grigorian, A. T.: Die Entwicklung der Hydrodynamik und Aerodynamik in den Arbeiten von N. E. Shikowski und S. A. Tschaplygin, Naturwissenschaften Technik und Medizin, No. 5, Leipzig, (1965), 39-62.

3. Pars, L. A.: An Introduction to the Calculus of Variations, Heinemann, London, (1962).

4. Rimrott, F. P. J.: Introductory Orbit Dynamics, Vieweg, Wiesbaden, (1989).

Quantity	Chaplygin's v_a, v_w	Kepler's μ, ε
Eccentricity	$\varepsilon = \frac{v_w}{v_a}$	ε
Gravitational attraction	$\mu = v_a^2 p = v_a h$	μ
Air speed	v_a	$v_a = \sqrt{\frac{\mu}{p}} = \frac{\mu}{h}$
Wind speed	v_w	$v_w = \varepsilon\sqrt{\frac{\mu}{p}} = \varepsilon\frac{\mu}{h}$
Semi-parameter	$p = \frac{h}{v_a}$	$p = \frac{h^2}{\mu}$
Specific angular momentum	$h = v_a p$	$h = \sqrt{\mu p}$
Kinetic energy	$T = \frac{1}{2}m\left(v_a^2 + 2v_a v_w \cos\theta + v_w^2\right)$	$T = \frac{1}{2}\mu m\left(\frac{2}{r} - \frac{1-\varepsilon^2}{p}\right)$
Potential energy	$V = -\frac{v_a^2 pm}{r} = -\frac{v_a h}{r}$	$V = -\frac{\mu m}{r}$
Total energy	$E = T + V = -\frac{v_a^2 m}{2}\left(1-\frac{v_w^2}{v_a^2}\right)$	$E = -\frac{\mu m}{2p}\left(1-\varepsilon^2\right)$
Areal velocity	$\dot{A} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}v_a p = \frac{1}{2}h$	$\dot{A} = \frac{1}{2}\sqrt{\mu p} = \frac{1}{2}h$
Area	$A = \pi ab = \frac{\pi p^2}{\left[1-\frac{v_w^2}{v_a^2}\right]^{3/2}}$	$A = \frac{\pi p^2}{\left[1-\varepsilon^2\right]^{3/2}}$
Time period	$\tau = \frac{2\pi p}{v_a\left[1-\frac{v_w^2}{v_a^2}\right]^{3/2}}$	$\tau = \frac{2\pi p}{\mu^{1/2}\left(1-\varepsilon^2\right)^{3/2}}$
Radial speed	$v_r = v_w \sin\theta$	$v_r = \frac{\mu}{h}\varepsilon\sin\theta$
Transverse speed	$v_t = v_a\frac{p}{r}$	$v_t = \frac{\sqrt{\mu p}}{r}$
Acceleration	$a_r = -\frac{v_a^2 p}{r^2} = -\frac{v_a h}{r^2}$	$a_r = -\frac{\mu}{r^2}$
Central force	$F_r = -\frac{m v_a h}{r^2}$	$F_r = -\frac{m\mu}{r^2}$

Table 1. Analogous quantities

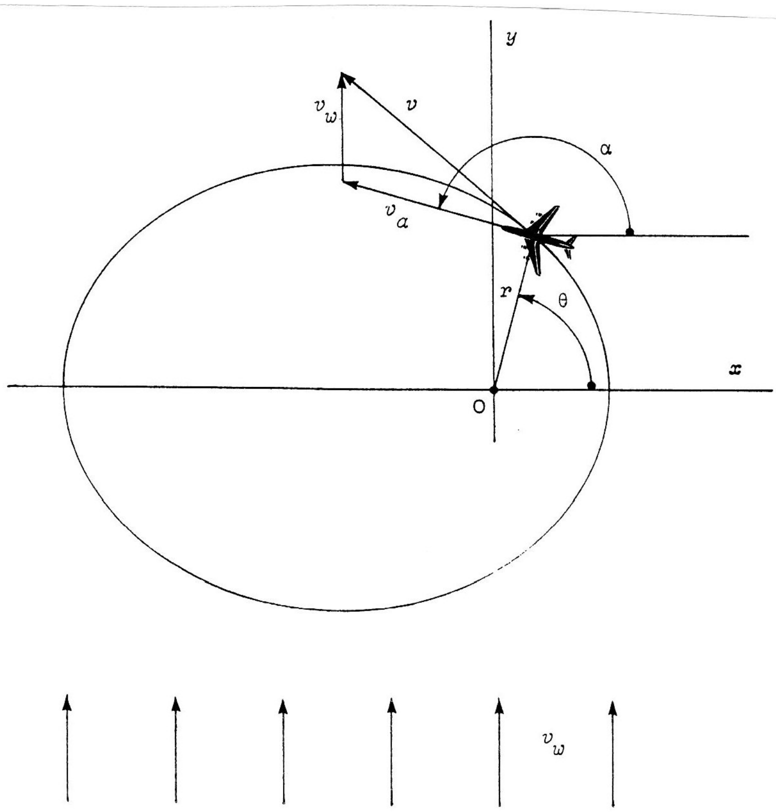

Figure 1. Chaplygin's problem

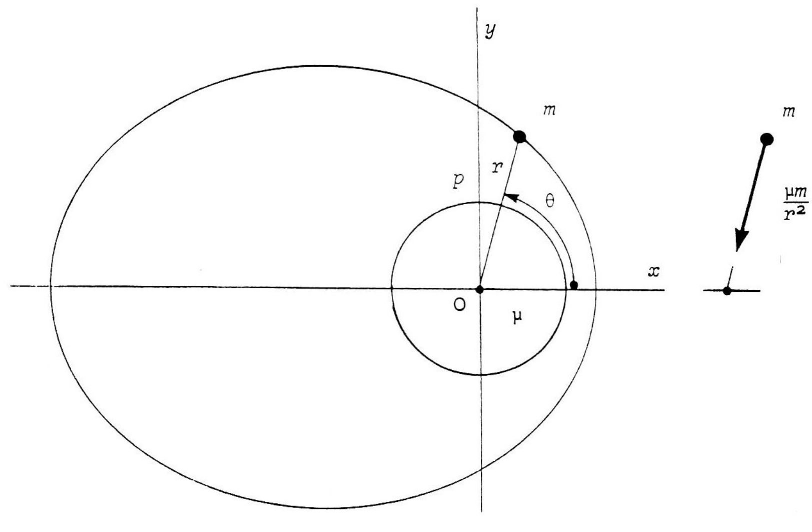

Figure 2. Kepler's problem

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