Př dokončení z posledního cvičení:

najděte $f = u + iv$ tak, aby byla diferencovatelná, pokud

$$u(x,y) = e^x(x\cos y - y\sin y)$$

$$\int y\sin y = -y\cos y + \sin y$$

$\Updownarrow$ per partes

$$u_x = e^x(x\cos y - y\sin y + \cos y)$$

$$v = \int u_x\,dy = e^x(x\sin y + \sin y) - e^x\int y\sin y\,dy =$$

$$= e^x(x\sin y + \sin y) - e^x(-y\cos y + \sin y) =$$

$$= e^x(x\sin y + \sin y + y\cos y - \sin y) = e^x(x\sin y + y\cos y) + g(x)$$

$$\frac{\partial v}{\partial x} = e^x(x\sin y + y\cos y + \sin y) + \frac{\partial g}{\partial x}$$

$$\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial g}{\partial x} = 0 \Rightarrow g = \text{konst} = C \in \mathbb{R}$$

Máme tedy

$$f(x,y) = e^x(x\cos y - y\sin y) + i\left[e^x(x\sin y + y\cos y) + C\right]$$

$$= e^x\left[x(\cos y + i\sin y) + y(-\sin y + i\cos y)\right] + iC = \leftarrow |K = iC|$$

$$= e^x\left[x e^{iy} + iy(\cos y + i\sin y)\right] + K =$$

$$= e^x\left(x e^{iy} + iy e^{iy}\right) + K = e^x e^{iy}(x + iy) = z e^z + K$$

1) Ukažte, že Cauchyho-Riemannovy podmínky lze napsat jako

$$\frac{df}{dz^*} = 0$$

Řešení:

$$\frac{df}{dz^*} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*}$$

$$x = \frac{z+z^*}{2} \Rightarrow \frac{\partial x}{\partial z^*} = \frac{1}{2}$$

$$y = \frac{z-z^*}{2i} \Rightarrow \frac{\partial y}{\partial z^*} = \frac{i}{2}$$

$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0 \Leftrightarrow \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \Leftrightarrow \underline{i\frac{\partial f}{\partial y} - \frac{\partial f}{\partial y} = 0}$$

C-R podmínky, viz DÚ 3

2) Určete C-R podmínky pro funkci ve tvaru

$$f(x,y) = R(x,y)\, e^{i\phi(x,y)}$$

Řešení:

$$\frac{\partial f}{\partial x} = \frac{\partial R}{\partial x}e^{i\phi} + iRe^{i\phi}\frac{\partial \phi}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial R}{\partial y}e^{i\phi} + iRe^{i\phi}\frac{\partial \phi}{\partial y}$$

$$i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = e^{i\phi}\left(i\frac{\partial R}{\partial x} - R\frac{\partial \phi}{\partial x} - \frac{\partial R}{\partial y} - iR\frac{\partial \phi}{\partial y}\right) = 0 \Leftrightarrow$$

$$\Leftrightarrow \boxed{\frac{\partial R}{\partial x} = R\frac{\partial \phi}{\partial y} \quad \& \quad \frac{\partial R}{\partial y} = -R\frac{\partial \phi}{\partial x}}$$

3) Určete C-R podmínky pro funkci ve tvaru

$$f(r,\varphi) = R(r,\varphi)\, e^{i\phi(r,\varphi)}$$

Řešení: $u(r,\varphi) = R(r,\varphi)\cos\phi(r,\varphi)$, $v(r,\varphi) = R(r,\varphi)\sin\phi(r,\varphi)$

použijeme CRP v pol. souř., viz cvičení 24.3.

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial R}{\partial r}\cos\phi - R\sin\phi\frac{\partial \phi}{\partial r} \\ \frac{\partial v}{\partial \varphi} &= \frac{\partial R}{\partial \varphi}\sin\phi + R\cos\phi\frac{\partial \phi}{\partial \varphi} \end{aligned}\right\} \quad r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \varphi} \Rightarrow \left(r\frac{\partial R}{\partial r} - R\frac{\partial \phi}{\partial \varphi}\right)\cos\phi + \left(-rR\frac{\partial \phi}{\partial r} - \frac{\partial R}{\partial \varphi}\right)\sin\phi = 0$$

$$\left.\begin{aligned} \frac{\partial v}{\partial r} &= \frac{\partial R}{\partial r}\sin\phi + R\cos\phi\frac{\partial \phi}{\partial r} \\ \frac{\partial u}{\partial \varphi} &= \frac{\partial R}{\partial \varphi}\cos\phi - R\sin\phi\frac{\partial \phi}{\partial \varphi} \end{aligned}\right\} \quad \frac{\partial u}{\partial \varphi} = -r\frac{\partial v}{\partial r} \Rightarrow 0 = \left(\frac{\partial R}{\partial \varphi} + rR\frac{\partial \phi}{\partial r}\right)\cos\phi + \left(-R\frac{\partial \phi}{\partial \varphi} + r\frac{\partial R}{\partial r}\right)\sin\phi$$

označme:

$$a = r\frac{\partial R}{\partial r} - R\frac{\partial \phi}{\partial \varphi}, \quad b = rR\frac{\partial \phi}{\partial r} + \frac{\partial R}{\partial \varphi}$$

Pak máme soustavu rovnic

$$\begin{aligned} a\cos\phi - b\sin\phi &= 0 \\ b\cos\phi + a\sin\phi &= 0 \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Homogenní soustava má netriviální řešení $\Leftrightarrow$ determinant matice soustavy je roven nule.

Protože $\begin{vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{vmatrix} = \cos^2\phi + \sin^2\phi = 1$,

máme $a = b = 0$.

Tedy $\boxed{r\frac{\partial R}{\partial r} = R\frac{\partial \phi}{\partial \varphi}, \quad \frac{\partial R}{\partial \varphi} = -rR\frac{\partial \phi}{\partial r}}$

4) Najděte funkci $f(r,\varphi) = u(r,\varphi) + i\,v(r,\varphi)$ diferencovatelnou tak, aby
$$u(r,\varphi) = r\varphi\cos\varphi + r\ln r\sin\varphi$$

Řešení

$$u_\varphi = r\cos\varphi - r\varphi\sin\varphi + r\ln r\cos\varphi = -r\,v_r$$

$$v = \int v_r\,dr = \int -\frac{1}{r}u_\varphi\,dr = \int -\cos\varphi + \varphi\sin\varphi - \ln r\cos\varphi\;dr$$

$$= -r\cos\varphi + r\varphi\sin\varphi - \cos\varphi\int \ln r\,dr$$

$$= -\cancel{r\cos\varphi} + r\varphi\sin\varphi - r\ln r\cos\varphi + \cancel{r\cos\varphi} + g(\varphi)$$

$$\int \ln r\,dr \overset{\text{per partes}}{=} r\ln r - \int dr = r\ln r - r$$

$$= r\varphi\sin\varphi - r\ln r\cos\varphi + g(\varphi)$$

$$v_\varphi = r\ln r\sin\varphi + r\sin\varphi + r\varphi\cos\varphi + g'(\varphi) = r\,u_r$$

$$u_r = \varphi\cos\varphi + \ln r\sin\varphi + \sin\varphi$$

$$r\,u_r = r\varphi\cos\varphi + r\ln r\sin\varphi + r\sin\varphi \quad\Rightarrow\quad g = C \in \mathbb{R}$$

$$\Rightarrow\; v(r,\varphi) = -r\ln r\cos\varphi + r\varphi\sin\varphi + C$$

$$f(z) = r\varphi\cos\varphi + r\ln r\sin\varphi + i\left(-r\ln r\cos\varphi + r\varphi\sin\varphi + C\right)$$

$$= r\varphi(\cos\varphi + i\sin\varphi) + r\ln r(\sin\varphi - i\cos\varphi) + iC$$

$$= r\varphi e^{i\varphi} - i r\ln r\, e^{i\varphi} + iC =$$

$$= r e^{i\varphi}(\varphi - i\ln r) + iC = -iz\ln z + iC$$

Cauchyho-Riemannovy podmínky - souhrn

1) $f(x,y) = u(x,y) + i\,v(x,y)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

2) $f(r,\varphi) = u(r,\varphi) + i\,v(r,\varphi)$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \varphi} \quad , \quad \frac{1}{r}\frac{\partial u}{\partial \varphi} = -\frac{\partial v}{\partial r}$$

3) $f(x,y) = R(x,y)\, e^{i\phi(x,y)}$

$$\frac{\partial R}{\partial x} = R\frac{\partial \phi}{\partial y} \quad , \quad \frac{\partial R}{\partial y} = -R\frac{\partial \phi}{\partial x}$$

4) $f(r,\varphi) = R(r,\varphi)\, e^{i\phi(r,\varphi)}$

$$\frac{\partial R}{\partial r} = \frac{1}{r}\cdot R\,\frac{\partial \phi}{\partial \varphi} \quad , \quad \frac{1}{r}\frac{\partial R}{\partial \varphi} = -R\frac{\partial \phi}{\partial r}$$

Mnemotechnika:

a) derivace jsou "do kříže", tj. neopakují se funkce ani proměnné

b) pořadí: (x,y), (u,v), (r,φ), (R,ϕ)

c) $\dfrac{\partial\,\text{první}}{\partial\,\text{první}}$, $\dfrac{\partial\,\text{druhá}}{\partial\,\text{druhá}}$, $\dfrac{\partial\,\text{druhá}}{\partial\,\text{první}}$, $-\dfrac{\partial\,\text{první}}{\partial\,\text{druhá}}$ má mínus!

d) před $\dfrac{\partial}{\partial\varphi}$ stojí $\dfrac{1}{r}$, před $\dfrac{\partial\phi}{\partial\,\text{cokoliv}}$ stojí R