Statistická fyzika 20.5.2021

42. Bílý trpaslík

a) N .... počet nukleonů

$N_e = ?$

$^{12}_{6}C$, $^{16}_{8}O$

	C	O
protony:	6	8
neutrony	6	8
elektrony	6	8
	12 nukl.	16
	6 elek.	8

počet elektronů poloviční oproti počtu nukleonů

$$N_e = N/2$$

b) 3D plně degenerovaný ferm. plyn

$$n(\vec{p})\, d\vec{p} = \frac{V}{(2\pi\hbar)^3}\, g\, d\vec{p}$$

pro 3D: sférické: p, φ, ϑ

integrace přes φ, ϑ

$$n(p)\, dp = \frac{V}{(2\pi\hbar)^3}\, \underbrace{g}_{=2}\, p^2 4\pi\, dp$$

počet částic

$$\boxed{N = \int_0^{p_F} dp\, n(p) = \frac{V}{\pi^2 \hbar^3} \frac{p_F^3}{3}} \qquad (1)$$

$$E = \int_0^{p_F} dp\, \underbrace{\epsilon}_{\frac{p^2}{2m}}\, n(p) = \frac{V}{\pi^2 \hbar^3} \int_0^{p_F} dp\, \frac{p^2}{2m} \cdot p^2 =$$

$$= \frac{V}{2 m_e \pi^2 \hbar^3} \int_0^{p_F} dp\, p^4 = \boxed{\frac{V p_F^5}{10 m_e \pi^2 \hbar^3}} \qquad (2)$$

(2)

$$\frac{E}{N} = \frac{\dfrac{\cancel{V}\, p_F^{\cancel{5}\,2}}{10 m_e \cancel{\pi^2} \cancel{\hbar^3}}}{\dfrac{\cancel{V}\, \cancel{p_F^3}}{3 \cancel{\pi^2} \cancel{\hbar^3}}} = \frac{3 p_F^2}{10 m_e}. \qquad (3)$$

(1)

$$V = \frac{4}{3}\pi R^3$$

Fermiho hybnost z (1)

$$p_F = \left(\frac{3 N \pi^2 \hbar^3}{V}\right)^{1/3} = \sqrt[3]{\frac{3 N_e \pi^2}{V}}\, \hbar =$$

$$= \sqrt[3]{\frac{3 N \pi^2}{\frac{4}{3}\pi R^3}}\, \hbar = \sqrt[3]{\frac{9 N_e \pi}{4}}\, \frac{\hbar}{R} \qquad (4)$$

Dosadíme p_F do (3)

$$\frac{E}{N} = \frac{3}{10 m_e} \cdot \left(\frac{9 N_e \pi}{4}\right)^{2/3} \frac{\hbar^2}{R^2} \qquad (6)$$

$$\frac{E}{N_e} \propto R^{-2} \quad , \quad \frac{E}{N_e} \propto N_e^{2/3}$$

---

ii ultrarel. $E = pc$

$$N \text{ viz } (1) = \frac{V}{\pi^2 \hbar^3} \frac{p_F^3}{3}$$

$$E = \frac{V}{\pi^2 \hbar^3} \int_0^{p_F} dp\, p \cdot c \cdot p^2 =$$

$$= \frac{Vc}{\pi^2 \hbar^3} \frac{p_F^4}{4}$$

$$\frac{E}{N} = \frac{\cancel{\frac{Vc}{\pi^2 \hbar^3}} \frac{p_F^{\cancel{4}}}{4}}{\cancel{\frac{V}{\pi^2 \hbar^3}} \cancel{\frac{p_F^3}{3}}} = \frac{3c\, p_F}{4} = \frac{3}{4} p_F \cdot c =$$

$$= \frac{3}{4} \cdot c \cdot \left(\frac{9 N_e \pi}{4} \right)^{1/3} \frac{\hbar}{R} = \quad (7)$$

P_F viz (4) ✓

$$\frac{E}{N_e} \propto R^{-1}, \quad \frac{E}{N_e} \propto N_e^{1/3}$$

c) $$E_G = -\frac{Gm^2}{R} \qquad (5)$$

předp. hmotnost je dána hlavně nukleony

$$m = N \cdot u$$

dos. do (5)

$$E_G = -\frac{G N^2 u^2}{R} \rightarrow \frac{E_G}{N} = -\frac{G N u^2}{R} \qquad (8)$$

d) Celková energie $\bar{e}$ plynu

$$E = \text{kinetická} + \text{potenciální}$$

sčítat výsl. (6), nebo (7)

$$\boxed{\begin{aligned} N_\nu &= \frac{N_e}{2} \rightarrow N_\nu = 2N_e \\ N_\nu^2 &= 4N_e^2 \end{aligned}}$$

e^- se pohybují v grav. poli nukleonů, viz (8)

$$\frac{E}{N_e} = \frac{E_k}{N_e} + \frac{E_p}{N_e} = \frac{3}{10m_e}\left(\frac{9}{4}\pi\hbar^3\right)^{2/3}\frac{N_e^{2/3}}{R^2} - \frac{4GN_e v^2}{R}$$

Lok. extrém, tehdy $d/dR = 0$

$$\frac{d}{dR}\left(\frac{E}{N_e}\right) = 0 = \frac{3}{10m_e}\left(\frac{9}{4}\pi\hbar^3\right)^{2/3}\frac{\cancel{N_e^{2/3}}}{\cancel{R^3}}(-\cancel{2}) + \frac{\cancel{2}\,\cancel{4}GN_e^{1/3}v^2}{\cancel{R^2}}$$

$$\frac{3}{10m_e}\left(\frac{9}{4}\pi\hbar^2\right)^{2/3}\frac{1}{R} = 2GN_e^{1/3}v^2$$

$$R = \frac{3}{10m_e}\left(\frac{9}{4}\pi\hbar^2\right)^{2/3}\frac{1}{2GN_e^{1/3}v^2}$$

Hledání minim funkce

$$f = \frac{1}{R}(A - B)$$

vetrvare l.

$$R \propto N^{-1/3}$$

Dvě možná minima

Minima		
$\frac{E}{N_e} < 0$	$R \to 0^+$	$\to -\infty$
$\frac{E}{N_e} > 0$	$R \to \infty$	$\to 0$

$$\frac{E}{N_e} = \dots\dots = \frac{3}{4} c N_e^{1/3} \hbar \left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{R} - \frac{46 N_e v^2}{R} \quad (9)$$

$$\frac{d}{dR}\left(\frac{E}{N_e}\right) = 0 = -\frac{3}{4} c N_e^{1/3} \hbar \left(\frac{9\pi^2}{4}\right)^{1/3} \frac{1}{R^2} + 46 N_e v^2 \frac{1}{R^2} =$$

$$= \frac{N_e}{R^2}\left[\underbrace{46 v^2 - \frac{3}{4} c \frac{\hbar}{N_e^{2/3}} \left(\frac{9\pi^2}{4}\right)^{1/3}}_{0}\right]$$

$$\dots$$

z (9)

$$\frac{E}{N_e} = \frac{N_e}{R}\left[\frac{3}{4} \frac{c\hbar}{N_e^{2/3}} \left(\frac{9\pi}{4}\right)^{1/3} - 46 v^2\right] \quad (10)$$

odhad: N_e řádově větší než ostatní konstanty

$\Rightarrow$ zmenší první výraz

$\Rightarrow [\quad] < 0$

minimum $\frac{E}{N_e}$ nastává pro $R \to 0^+$

když $\frac{E}{N_e} \to -\infty$

e) z (10) $[\quad] = 0$

$$N_e^{2/3} = \frac{3}{4} \frac{c\hbar}{4Gu^2} \left(\frac{9\pi}{4}\right)^{1/3}$$

$$N_e = \left(\frac{1}{16} \frac{c\hbar}{Gu^2}\right)^{3/2} \left(\frac{9\pi}{4}\right)^{1/2} \qquad (11)$$

předp. hmotnost se soustřeďuje převážně v nukleonech

$$N_N = 2N_{el}$$

potom

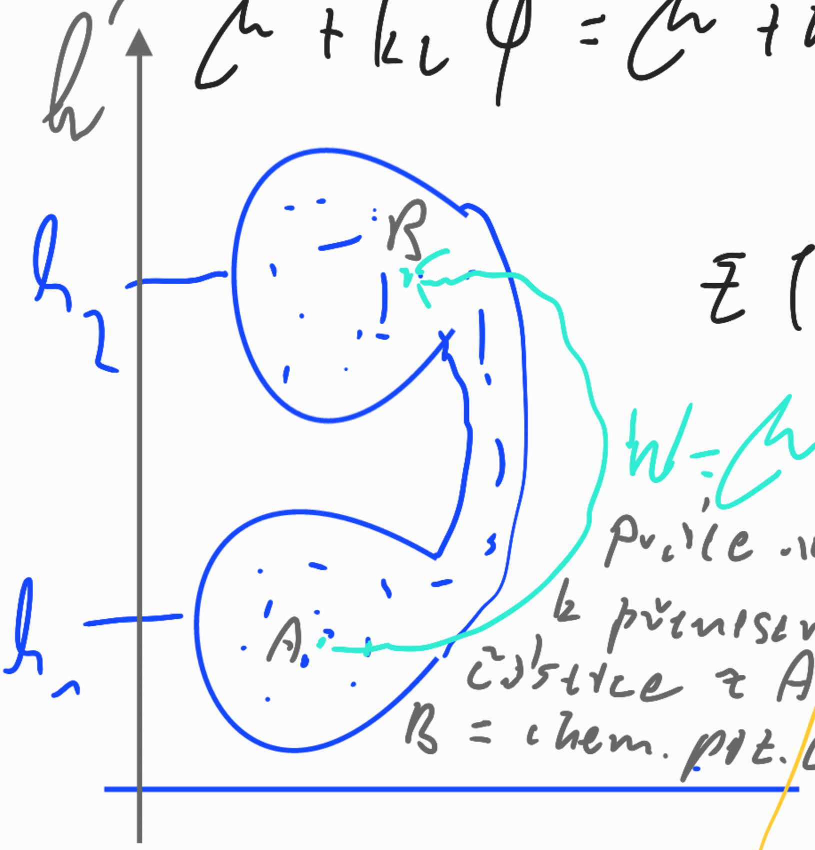
$$m_{krit.} = 2N_e v = \qquad (11)$$

$$= 2v \cdot \left(\frac{1}{16}\frac{ch}{Gv^2}\right)^{3/2}\left(\frac{5\pi}{4}\right)^{1/2}$$

---

13 μ rce pro chem. potenciál v tíhovém poli

$$\mu + k_v \varphi = \mu' + mc^2 + k_v \varphi = konst. \qquad (12)$$



$W = \mu$

Práce nutná k přenesení částice z A do B = chem. pot. μ

z (12)

$$\varphi = \frac{konst.}{k_v} - \frac{\mu}{k_v}$$

$$\Delta\varphi = 4\pi G\rho$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) + \underbrace{\dots}_{0} = -4\pi G \rho$$

sfér.

sym.

$\rho = k u n$

hustota

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{ku}\frac{\partial \mu^{ch}}{\partial r}\right) = 4\pi G k \cdot u \cdot n \,/\cdot ku$$

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \mu^{ch}}{\partial r}\right) = -4\pi G k^2 u^2 n} \quad (13)$$

---

44. $\int$ integrální tvar rce pro chem. p.

$$\left.\frac{d\mu^{ch}}{dr}\right|_{r=0} = 0$$

Řeš.: (13) vynásobíme r^2

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi_h}{\partial r}\right) = -4\pi r^2 G k_r^2 n \Big/ \int^R$$

$$\underbrace{\int_0^R dr \frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi_h}{\partial r}\right)}_{\frac{\partial}{\partial r} = \frac{d}{dr}\qquad C} = -4\pi G k_r^2 \underbrace{\int_0^R dr\, n r^2}_{P}$$

$$r^2 \frac{\partial \phi_h}{\partial r}\Big|_{r=R} - \underbrace{r^2 \frac{\partial \phi}{\partial r}\Big|_{r=0}}_{0} = R^2 \frac{\partial \phi_h}{\partial r}\Big|_{r=R}$$

$$P: \underbrace{4\pi}_{\int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin\vartheta} \cdot G k_r \int_0^R dr\, \overbrace{k_r \cdot n}^{\rho}\, r^2 =$$

$$M = G k_r M$$

$$\boxed{R^2 \frac{\partial \phi}{\partial r}\bigg|_{r=R} = -GkuM}$$