Introduction to Cosmology

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Abstract: This paper contains notes devoted to the Introduction to cosmology

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1. Basic Principles

1.1 Units

We mostly use the natural system of units where the Planck constant, speed of light and the Boltzman constant are equal to one

$$\hbar = c = k_B = 1 \ . \tag{1.1}$$

Then the mass M, energy E and temperature T have the same dimensions since

$$[E] = [Mc^2] = [M] \tag{1.2}$$

and also we have

$$[E] = [k_B T] = [T] = [M] \ . \tag{1.3}$$

Time t and length l have in natural system dimension $[M]^{-1}$ as follows from the fact that

$$[E] = [\hbar\omega] = [\omega] = [t^{-1}] \tag{1.4}$$

so that $[t] = [M]^{-1}$. In the same way we have

$$[l] = [ct] = [t] = [M]^{-1} \ . \tag{1.5}$$

It is useful to know coeficients that relate various units

Quantity	SI dimensions	Natuaral dimensions	Conversions
mass	kg	M	$1GeV = 1.8 \times 10^{-27} kg$
length	m	M^{-1}	$1GeV^{-1} = 0.197 \times 10^{-15} m$
time	s	M^{-1}	$1GeV^{-1} = 6.58 \times 10^{-25} s$
energy	$kg \cdot m^2 \cdot s^{-2}$	M	$1GeV = 5.39 \times 10^{-19} kg \cdot m \cdot s^{-1}$
momentum	$kg \cdot m \cdot s^{-1}$	M	$1GeV = 5.39 \times 10^{-19} kg \cdot m \cdot s^{-1}$
velocity	$m \cdot s^{-1}$		$1 = 2.998 \times 10^8 m \cdot s^{-1}$
cross section	m^2	M^{-2}	$1GeV^{-2} = 0.389 \times 10^{-31} m^2$
force	$kg \cdot m \cdot s^{-2}$	M^2	$1GeV^2 = 8.19 \times 10^5$Newton

The traditional unit of length in cosmology is Megaparsec

$$1\ Mpc = 3.1 \times 10^{22} m\ . \tag{1.6}$$

It is interesting to mention the several units of length that are used in astronomy. Besides the metric system in use are the astronomical unit ($a.u.$) which is the average distance from the Earth to the Sun

$$1\ a.u. = 1.5 \times 10^{11} m \tag{1.7}$$

Further, there is a light year, the distance that a photon travels in one year

$$1 \text{ year} = 3.16 \times 10^7 s\ , 1 \text{ light year} = 0.95 \times 10^{16} m \tag{1.8}$$

parsec (pc)-distance from which an object of size $1a.u.$ is seen at angle $1arc$ second

$$1\ pc = 2.1 \cdot 10^5 a.u. = 3.3 \text{ light year} = 3.1 \times 10^{16} m \tag{1.9}$$

It is instructive to give distances of various objects expressed in above units.

$10a.u.$ is the average disance to Saturn, $30a.u.$ is the same for Pluto, $100a.u.$ is the estimate of the maximum distance which can be reached by solar wind (particles emitted by the Sun). The nearest stars-Proxima and Alpha Centauri are at $1.3pc$ from the Sun. The distance to Arcturus and Capella is more than $10pc$, the distances to Canopus and Betelgeuse are about $100pc$ and $200pc$ respectively. Crab Nebula-the remnant of supernova is $2kpc$ away from us.

The next point on the scale of distance is $8kpc$. This is the distance from the Sun to the center of our Galaxy. Our Galaxy is of spiral type, the diameter of its disc is about $30kpc$ and the thickness of the disc is about $250pc$. The distance to the nearest dwarf galaxies that are satelites of our Galaxy is about $30kpc$. Fifteen of these satellites are known, the largest of them are Large and Small Magellanic Clouds are about $50kpc$ away. It is also interesting to note that only eight Milky Way satellites were known by 1994.

The mass density of the usual matter in usual (not dwarf) galaxies is about 10^5 higher than the average over Universe.

The nearest usual galaxy-the spiral galaxy $M31$ in Andromeda constellation- is $800kpc$ away from the Milky Way. Another nearby galaxy is in Triangulum constellation. Our Galaxy together with Andromeda and Triangulum galaxies , their satelites and other 35 smaller galaxies constitute the Local Group which is the gravitationally bound object consisting of about 50 galaxies.

The next scale is the size of clusters of galaxies which is $1-3Mpc$ Rich clusters contain thounsands of galaxies. The mass density in clusters exceeds the average density over the Universe by a factor of a hundred and even sometimes a thousand. The distance to the center of the nearest cluster, which is the Vigo constellation, is about $15Mpc$. Clusters of galaxies are the largest gravitationally bound systems in the Universe.

1.2 Gravitational Field Equations

As we know in General Relativity (GR) the metric tensor is dynamical field and the equations of GR arise as extremum conditions for the action functional. The princip equivalence means that all equations has to have the same form in all reference frames. In other words we require that the action function has to be same in all reference frames which means that the action is scalar. Since the action is given as the integral over time of the Lagrangian we find also that the Lagrangian has to be given as the integral over space section of the spacetime. In summary we postulate thath the gravity action has the form

$$S_{gr} = \int d^4x \sqrt{-g} \mathcal{L}_{gr} \ , \tag{1.10}$$

where the Lagrangian density $\mathcal{L}_{gr}(x)$ transforms as under coordinate transformations $x'^{\mu} = x^{\mu}(x)$

$$\mathcal{L}'(x') = \mathcal{L}(x) \tag{1.11}$$

and due to the fact that $d^4x'\sqrt{-g'(x')} = d^4x\sqrt{-g(x)}$ we really see that S_{gr} does not change under diffeomorphism transformations.

The simplest possibility is to take the Lagrangian density to be equal to constant $\mathcal{L} = -\Lambda$ so that

$$S_{\Lambda} = -\Lambda \int d^4x \sqrt{-g} \ . \tag{1.12}$$

However this action does not contain the time derivatives of the metric and hence the dynamics that would follow from this action is trivial. For that reason we should search more complicated form of the Lagrangian density.

The Lagrange density is a tensor density, which can be written as $\sqrt{-g}$ times a scalar that is function of the metric and its derivatives. The question is the form of given scalar. Since we know that the metric can be set equal to its canonical form and its first derivatives set to zero at any one point, any nontrivial scalar must involve at least second derivatives of the metric. The Riemann tensor is of course made from second derivatives of the metric, and we argued earlier that the only independent scalar we could construct from the Riemann tensor was the Ricci scalar R. What we did not show, but is nevertheless true, is that any nontrivial tensor made from the metric and its first and second derivatives can be expressed in terms of the metric and the Riemann tensor. Therefore, the only independent scalar constructed from the metric, which is no higher than second order in its derivatives, is the Ricci scalar. Hilbert figured that this was therefore the simplest possible choice for a Lagrangian, and proposed

$$\mathcal{L}_H = \sqrt{-g}R \ . \tag{1.13}$$

The equations of motion should come from varying the action with respect to the metric. In fact let us consider variations with respect to the inverse metric $g^{\mu\nu}$, which are slightly easier but give an equivalent set of equations. Using $R = g^{\mu\nu}R_{\mu\nu}$, in general we will have

$$\begin{aligned} \delta S &= \int d^n x \left[\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + R\delta\sqrt{-g}\right] \\ &= (\delta S)_1 + (\delta S)_2 + (\delta S)_3 \ . \end{aligned} \tag{1.14}$$

The second term $(\delta S)_2$ is already in the form of some expression times $\delta g^{\mu\nu}$; let's examine the others more closely.

Recall that the Ricci tensor is the contraction of the Riemann tensor, which is given by

$$R^{\rho}{}_{\mu\lambda\nu} = \partial\lambda\Gamma^{\lambda}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - (\lambda \leftrightarrow \nu) \ . \tag{1.15}$$

We perform the variation of the Riemann tensor in such a way that we firstly perform variation of the connection coefficients and then we substitute into this expression. In fact, after some calculations we find the variation of the Riemann tensor in the form

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_{\lambda}(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_{\nu}(\delta\Gamma^{\rho}_{\lambda\mu}) \ . \tag{1.16}$$

Therefore, the contribution of the first term in (1.14) to δS can be written

$$\begin{aligned} (\delta S)_1 &= \int d^4x\sqrt{-g}\ g^{\mu\nu}\left[\nabla_{\lambda}(\delta\Gamma^{\lambda}_{\nu\mu}) - \nabla_{\nu}(\delta\Gamma^{\lambda}_{\lambda\mu})\right] \\ &= \int d^4x\sqrt{-g}\ \nabla_{\sigma}\left[g^{\mu\sigma}(\delta\Gamma^{\lambda}_{\lambda\mu}) - g^{\mu\nu}(\delta\Gamma^{\sigma}_{\mu\nu})\right] \ , \end{aligned} \tag{1.17}$$

where we have used metric compatibility. However the integral above is an integral with respect to the natural volume element of the covariant divergence of a vector; by Stokes's theorem, this is equal to a boundary contribution at infinity which we can set to zero by making the variation vanish at infinity. Therefore this term does not contribute to the total variation.

In order to calculate the $(\delta S)_3$ term we have to use the variation

$$\delta(g^{-1}) = \frac{1}{g}g_{\mu\nu}\delta g^{\mu\nu} \ . \tag{1.18}$$

and consequently

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \ . \tag{1.19}$$

If we now return back to (1.14), and remembering that $(\delta S)_1$ does not contribute, we find

$$\delta S = \int d^4x\sqrt{-g}\ \left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\delta g^{\mu\nu} \ . \tag{1.20}$$

However this should vanish for arbitrary variations and consequently we derive Einstein's equations in vacuum:

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \ . \tag{1.21}$$

However we would like to get the non-vacuum field equations as well. In other words we consider an action of the form

$$S = \frac{1}{8\pi G}S_H + S_M \ , \tag{1.22}$$

where S_M is the action for matter, and we have presciently normalized the gravitational action (although the proper normalization is somewhat convention-dependent). Following through the same procedure as above leads to

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0 \ , \tag{1.23}$$

and we recover Einstein's equations if we set

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \ . \tag{1.24}$$

In fact (1.24) turns out to be the best way to define a symmetric energy-momentum tensor.

Einstein's equations may be thought of as second-order differential equations for the metric tensor field $g_{\mu\nu}$. There are ten independent equations (since both sides are symmetric two-index tensors), which seems to be exactly right for the ten unknown functions of the metric components. However, the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ which we prove below represents four constraints on the functions $R_{\mu\nu}$, so there are only six truly independent equations. In fact this is appropriate, since if a metric is a solution to Einstein's equation in one coordinate system x^μ it should also be a solution in any other coordinate system $x^{\mu'}$. This means that there are four unphysical degrees of freedom in $g_{\mu\nu}$ (represented by the four functions $x^{\mu'}(x^\mu)$), and we should expect that Einstein's equations only constrain the six coordinate-independent degrees of freedom.

It is important to stress that as differential equations, these are extremely complicated; the Ricci scalar and tensor are contractions of the Riemann tensor, which involves derivatives and products of the Christoffel symbols, which in turn involve the inverse metric and derivatives of the metric. Furthermore, the energy-momentum tensor $T_{\mu\nu}$ will generally involve the metric as well. The equations are also nonlinear, that implies that two known solutions cannot be superposed to find a third. It is therefore very difficult to solve Einstein's equations in any sort of generality. Then in order to solve them we have to perform some simplifying assumptions. The most popular sort of simplifying assumption is that the metric has a significant degree of symmetry, and we will talk later on about how symmetries of the metric make life easier.

We are mainly interested in the existence of solutions to Einstein's equations in the presence of "realistic" sources of energy and momentum. The most common property that is demanded of $T_{\mu\nu}$ is that it represent positive energy densities — no negative masses are allowed. In a locally inertial frame this requirement can be written as $\rho = T_{00} \geq 0$. We write it in the coordinate-independent notation as

$$T_{\mu\nu}V^{\mu}V^{\nu} \geq 0 \ , \qquad \text{for all timelike vectors } V^{\mu} \ . \tag{1.25}$$

This is known as the **Weak Energy Condition**, or WEC. It seems like a reasonable requirement however it is very restrictive. Indeed it is straightforward to show that there are many examples of the classical field theories which violate the WEC, and almost impossible to invent a quantum field theory which obeys it. Nevertheless, it is legitimate to assume that the WEC holds in most cases and it is violated in some extreme conditions. (There are also stronger energy conditions, but they are even less true than the WEC, and we won't dwell on them.)

An important property of the energy momentum tensor is that it is conserved. In the flat background the conservation equation takes the form

$$\partial_{\mu}T^{\mu\nu} = 0 \ , \tag{1.26}$$

where the first equation $\partial_{\mu}T^{\mu i} = 0$ expresses the conservation of the energy density while the remaining three equations $\partial_{\mu}T^{\mu i} = 0$ defines the conservation of the momentum density. In general relativity the conservation equation takes the form

$$\nabla_{\mu}T^{\mu\nu} = 0 \ . \tag{1.27}$$

This equation can be proved using the equation of motion for the metric when we apply the covariant derivative on both sides of this equation

$$\nabla^{\mu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 8\pi G\nabla^{\mu}T_{\mu\nu} \ . \tag{1.28}$$

We show that the left side of this equation is *identically zero*. Note that generally the matter fields do not have to be on shell since this equation follows from the variation of the action with respect to the metric. To see this we recall the Bianchi identity for the Riemann tensor

$$\nabla_\rho R^\lambda{}_{\sigma\mu\nu} + \nabla_\nu R^\lambda{}_{\sigma\rho\mu} + \nabla_\mu R^\lambda{}_{\sigma\nu\rho} = 0 \ . \tag{1.29}$$

Now we contract λ and μ indices and by definition $R^\mu{}_{\sigma\mu\nu} = R_{\sigma\nu}$ we obtain the identity

$$\nabla_\rho R_{\sigma\nu} - \nabla_\nu R_{\rho\sigma} + \nabla_\lambda R^\lambda{}_{\sigma\nu\rho} = 0 \ . \tag{1.30}$$

Then we contract this equation with $g^{\rho\sigma}$ and we obtain

$$0 = \nabla_\rho R^\rho{}_\nu - \nabla_\nu R + \nabla^\lambda R_{\lambda\nu} = 2\nabla^\mu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0 \ . \tag{1.31}$$

which implies that the covariant conservation law of the stress energy-tensor is a necessary condition for the consistency of the Einstein equation.

On the other hand the stress energy tensor is determined by the matter action. Clearly when we search the extremum of the action we perform the variation of the action with respect to the matter fields so that the energy momentum tensor should be conserved as the consequence of the matter equations of motions as well. Alternatively, we can presume the evolution of the matter fields on the fixed background and in this case the energy-momentum tensor should be conserved as well.

To proceed note that the matter action is diffeomorphism invariant so that the conservation of the energy momentum tensor should follow from the invariance of the action under general diffeomorphism transformation. In fact, under transformation

$$x'^\mu = x^\mu + \xi^\mu \ . \tag{1.32}$$

Then

$$\begin{aligned} g'^{\mu\nu}(x') &= g^{\rho\sigma} \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} \Rightarrow \\ g'^{\mu\nu}(x') &= g^{\mu\nu}(x) + g^{\nu\lambda}(x) \partial_\lambda \xi^\mu + \partial_\lambda x^\mu g^{\lambda\nu}(x) \end{aligned} \tag{1.33}$$

If we expand

$$g'^{\mu\nu}(x') = g'^{\mu\nu}(x+\xi) = g'^{\mu\nu}(x) + \partial_\lambda g'^{\mu\nu} \xi^\lambda = g'^{\mu\nu}(x) + \partial_\lambda g^{\mu\nu} \xi^\lambda \tag{1.34}$$

we find the variation $g^{\mu\nu}$ as

$$\delta g^{\mu\nu}(x) = g'^{\mu\nu}(x) - g^{\mu\nu}(x) = -\partial_\lambda g^{\mu\nu}(x) \xi^\lambda + g^{\mu\lambda} \partial_\lambda \xi^\nu + \partial_\lambda \xi^\mu g^{\lambda\nu} \ . \tag{1.35}$$

Now we proceed to the transformation property of the matter fields. Their form depends on the character of these fields, whether they are scalars, vectors,..... For example, in case of the scalar field we find

$$\phi'(x') = \phi(x) \Rightarrow \phi'(x) - \phi(x) = -\partial_\lambda \phi \xi^\lambda \tag{1.36}$$

Since the action is invariant under the diffeomorphism invariance we obtain

$$\delta_\xi S_m = \frac{1}{2}\int d^4x\sqrt{-g}T_{\mu\nu}(\nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu) + \int d^4x\sqrt{-g}\frac{\delta\mathcal{L}_m}{\delta\psi}\delta\psi_\xi = 0 \ , \tag{1.37}$$

where we also used the fact that the variation of the metric can be written as

$$g'^{\mu\nu} - g^{\mu\nu} = \nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu \tag{1.38}$$

Note that the equation (1.37) has to be zero of shell. Let us now presume that the matter field equations are satisfied which implies that the second term in (1.37) vanishes. Then using integration by parts we can rewrite (1.37) into the form

$$\delta_\xi S_m(on\ shell) = -\int d^4x\sqrt{-g}\xi^\mu\nabla^\mu T_{\mu\nu} = 0 \tag{1.39}$$

that using the fact that ξ^μ is arbitrary implies the conservation of the stress energy tensor.

We continue with the study of the Einstein equations where we now discuss the possibility of the introduction of a cosmological constant. In order to introduce it we add it to the conventional Hilbert action. We therefore consider an action given by

$$S = \int d^4x\sqrt{-g}(R - 2\Lambda) \ , \tag{1.40}$$

where Λ is some constant. The resulting field equations are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \ , \tag{1.41}$$

and of course there would be an energy-momentum tensor on the right hand side if we had included an action for matter. Λ is the cosmological constant. In order to find its meaning it is convenient to move the additional term in (1.41) to the right hand side, and think of it as a kind of energy-momentum tensor, with $T_{\mu\nu} = -\Lambda g_{\mu\nu}$ (it is automatically conserved by metric compatibility). Then Λ can be interpreted as the "energy density of the vacuum," a source of energy and momentum that is present even in the absence of matter fields. This interpretation is important because quantum field theory predicts that the vacuum should have some sort of energy and momentum. In ordinary quantum mechanics, an harmonic oscillator with frequency ω and minimum classical energy $E_0 = 0$ upon quantization has a ground state with energy $E_0 = \frac{1}{2}\hbar\omega$. A quantized field can be thought of as a collection of an infinite number of harmonic oscillators, and each mode contributes to the ground state energy. The result is of course infinite, and must be appropriately regularized, for example by introducing a cutoff at high frequencies. The final vacuum energy, which is the regularized sum of the energies of the ground state oscillations of all the fields of the theory, has no good reason to be zero and in fact would be expected to have a natural scale

$$\Lambda \sim m_P^4 \ , \tag{1.42}$$

where the Planck mass m_P is approximately 10^{19} GeV, or 10^{-5} grams. Observations of the universe on large scales allow us to constrain the actual value of Λ, which turns out to be smaller than (1.42) by at least a factor of 10^{120}. This is the largest known discrepancy between theoretical estimate and observational constraint in physics, and convinces many people that the "cosmological constant problem" is one of the most important unsolved problems today. On the other hand the observations do not tell us that Λ is strictly zero, and in fact allow values that can have important consequences for the evolution of the universe.

1.3 Basic principles of Cosmology

In this section we review basic facts about classical cosmology, following mainly [3]. There are many reviews available on hep-th, see for example [4, 5, 6] [1]. Contemporary cosmological modes are based on the idea that the Universe is pretty much the same everywhere-the idea known as **Copernican principle**. It is clear that this principle can be applied on the large scales only where local variations of density is averaged over. In other words, the Universe is spatially homogeneous and isotropic on the largest scales. Since these claims need more explanation let us pause in our explanation of cosmology and give some more precise definition of mathematical claims given above.

1.4 Map of Manifolds

Since we do not have enough time with explanation of the notion of manifold we presume that reader has enough knowledge regarding this point.

Let M and N be manifolds (generally with different dimensions) and let $\phi : M \rightarrow N$ be a map. In a natural manner, ϕ "pulls back" a function $f : N \rightarrow R$ on N to the function $f \circ \phi \rightarrow M \rightarrow R$ that is derived by composing f with ϕ. Similarly, in a natural way, ϕ maps tangent vectors at $p \in M$ to tangent vectors at $\phi(p) \in N$. In other words it defines ma $\phi^* : V_p \rightarrow V_{f(p)}$ in following way: For $V \int V_p$ we define $\phi^*(v)$ by

$$(\phi^*(v))(f) = v(f \circ \phi) \tag{1.43}$$

for all smooth $f : N \rightarrow R$. It is easy to see that $\phi^* v$ satisfies the properties of tangent vector at $\phi(p)$. Further, in the coordinate bases of a coordinate system (x^ν) at p and a coordinate system (y^μ) at $\phi(p)$ the upper expression takes the form

$$\begin{aligned}
& w^\mu(y)\frac{\partial}{\partial y^\mu} f(y) = v^\nu(x)\frac{\partial}{\partial x^\nu} f((\phi(x))) = v^\nu(x)\frac{\partial f(y)}{\partial y^\mu}\frac{\partial y^\mu}{\partial x^\nu} \Rightarrow \\
& w^\mu(\phi(x)) = v^\nu(x)\frac{\partial y^\mu}{\partial x^\nu} \ , \quad (\phi^* v)^\mu \equiv w^\mu \ .
\end{aligned} \tag{1.44}$$

[1] Our metric signature is $-+++$. We use units $\hbar = c = 1$ and define the reduced Planck mass by $M_p = (8\pi G)^{-1/2} \approx 10^{18} GeV$.

In the same way we can use ϕ to "pull back" one forms at $\phi(p)$ to one forms at p. We define the map ("pull back") $\phi_* : V^*_{\phi(p)} \to V^*_p$ by requiring that for $v \in V_p$

$$(\phi_*\omega)_\mu v^\mu = \omega_\nu (\phi^* v)^\nu \ , \tag{1.45}$$

where we used tensor notation. Using the definition of the map ϕ^* given in (1.44) we easily get

$$(\phi_*\omega)_\mu = \omega_\nu \frac{y^\nu}{\partial x^\mu} \ . \tag{1.46}$$

We can easily extend the action of ϕ_* to map tensors of type $(0, l)$ at $\phi(p)$ to tensors of type $(0, l)$ at p by

$$(\phi_* T)_{\mu_1 \dots \mu_l} v_1^{\mu_1} \dots v_l^{\mu_l} = T_{\mu_1 \dots \mu_l} (\phi^* v_1)^{\mu_1} \dots (\phi^* v_l)^{\mu_l} \ . \tag{1.47}$$

In the same way we can extend the action of ϕ^* to map tensors of type $(k, 0)$ at p to tensors of type $(k, 0)$ at $\phi(p)$ by

$$(\phi^* T)^{\mu_1 \dots \mu_k} (\omega_1)_{\mu_1} \dots (\omega_k)_{\mu_k} = T^{\mu_1 \dots \mu_k} (\phi_* \omega_1)_{\mu_1} \dots (\phi_* \omega_k)_{\mu_k} \tag{1.48}$$

If $\phi : M \to M$ is diffeomorphism and T is a tensor field on M we can compare T with $\phi^* T$. If $\phi^* T = T$ then even though we have moved T via ϕ it is still the same. In other words ϕ is a symmetry transformation for the tensor field T. In the case of the metric $g_{\mu\nu}$ a symmetry transformation-a diffeomorphism ϕ such that

$$(\phi^* g)_{\mu\nu} = g_{\mu\nu}$$

is called an isometry.

Let us now return to our explanation of basic principles of cosmology. Our first task is to formulate precisely the mathematical meaning of this assumption. The evidence comes from the smoothness of the temperature of the cosmic microwave background. In other words, given any two points p and q there is an isometry which takes p into q. We must mention that there is no necessary relationship between homogeneity and isotropy; a manifold can be homogeneous but nowhere isotropic (such as $R \times S^2$ in the usual metric) or it can be isotropic around a point without being homogeneous (such as a cone, which is isotropic around its vertex but certainly not homogeneous). On the other hand, if a space is isotropic *everywhere* then it is homogeneous. On the other hand it should be pointed that, in general, at each point, at most one observer can see the universe as isotropic. For example, if ordinary matter fills the universe, any observer in motion relative to the matter must see an anisotropic velocity distribution of the matter. With this fact in mind we have to give precise formulation of the notion of isotropy than the clam that *Isotropy is the claim that the Universe looks the same in all directions.*: A spacetime is said to be (spatially) isotropic at teach point if there exists a congruence of time-like curves (observes) with tangent vectors denoted u^μ filling the spacetime and