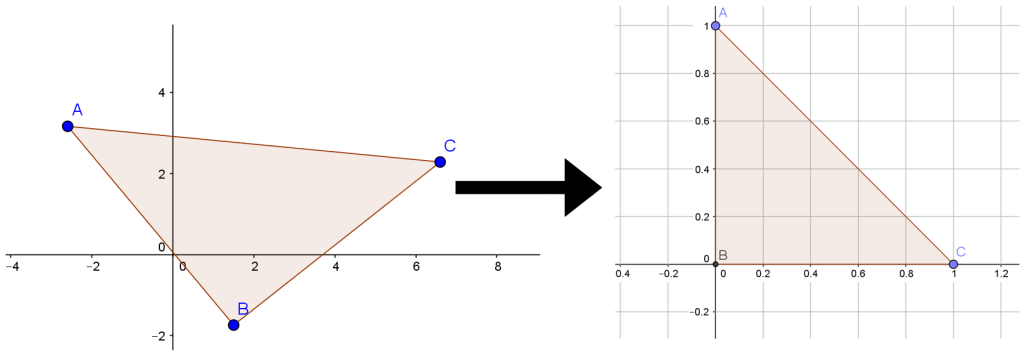
# Integral nad trojuholnikom

$$\int\int \mathrm{d}S f(x,y) \tag{1}$$

## Myslienka zjedodusenia




Chcem integrovat v suradniciach v ktorych dve strany trojuholnika lezia na suradniciovych osiach, tj. mat integral:

$$\int_0^1 \mathrm{d}u \int_0^{1-u} f(u,v)\mathrm{d}v, \tag{2}$$

To je mozne, ked novy suradnicovy system bude mat nulu v bode napr. B a osi $u, v$ budu v smere vektorov $\vec{BC}$ a $\vec{BA}$ naskalovane tak, ze body A a C budu lezat na jednickach prislusnych os.

$$x = b_x + (a_x - b_x)v + (c_x - b_x)u \tag{3}$$
$$y = b_y + (a_y - b_y)v + (c_y - b_y)u \tag{4}$$

Transformacne rovnice vyzeraju, ze splnuju kriteria.

Jakobian transformacie je:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} c_x - b_x & a_x - b_x \\ c_y - b_y & a_y - b_y \end{vmatrix} = (c_x - b_x)\left(a_y - b_y\right) - (a_x - b_x)\left(c_y - b_y\right) \tag{5}$$

Vysledny integral je teda vo forme:

$$|(c_x - b_x)\left(a_y - b_y\right) - (a_x - b_x)\left(c_y - b_y\right)| \int_0^1 \mathrm{d}u \int_0^{1-u} f(x(u,v), y(u,v))\mathrm{d}v \tag{6}$$

## Aplikacie na momenty

Pomocne vysledky integralov:

$\int_0^1 \mathrm{d}u \int_0^{1-u} u^i v^j \mathrm{d}v$

| $^i/_j$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1/2 | 1/6 | 1/12 |
| 1 | 1/6 | 1/24 | |
| 2 | 1/12 | | |

Pre $f(x,y) = x$:

$$\int\int x \mathrm{d}S = |J| \int_0^1 \mathrm{d}u \int_0^{1-u} \left(b_x + (a_x - b_x)\, v + (c_x - b_x)\, u\right) \mathrm{d}v = \tag{7}$$

$$|J| \left(\frac{1}{2} b_x + \frac{1}{6}\left(a_x - b_x\right) + \frac{1}{6}\left(c_x - b_x\right)\right) = |J| \left(\frac{1}{6} a_x + \frac{1}{6} c_x + \frac{3-2}{6} b_x\right) = \tag{8}$$

$$\frac{1}{6}|J|\left(a_x + b_x + c_x\right) \tag{9}$$

Pre $f(x,y) = x^2$:

$$x^2 = (b_x + (a_x - b_x)v + (c_x - b_x)u)^2 = \tag{10}$$

$$b_x^2 + (a_x - b_x)^2 v^2 + (c_x - b_x)^2 u^2 + 2b_x(a_x - b_x)v + 2b_x(c_x - b_x)u + 2(a_x - b_x)(c_x - b_x)uv \tag{11}$$

$$\int\int x^2 \mathrm{d}S = \tag{12}$$

$$|J| \left(\frac{1}{2} b_x^2 + (a_x - b_x)^2 \frac{1}{12} + (c_x - b_x)^2 \frac{1}{12} + 2b_x(a_x - b_x)\frac{1}{6} + 2b_x(c_x - b_x)\frac{1}{6} + 2(a_x - b_x)(c_x - b_x)\frac{1}{24}\right) = \tag{13}$$

$$|J|\frac{1}{12}\left(6b_x^2 + a_x^2 - 2a_x b_x + b_x^2 + c_x^2 - 2c_x b_x + b_x^2 + 4b_x a_x - 4b_x^2 + 4b_x c_x - 4b_x^2 + a_x c_x - a_x b_x - b_x c_x + b_x^2\right) = \tag{14}$$

$$|J|\frac{1}{12}\left(b_x^2 + a_x^2 + c_x^2 + b_x a_x + b_x c_x + a_x c_x\right) \tag{15}$$