$$y' + p(x)y = f(x)$$

$$\int \frac{dy}{y} = \int a(x)\,dx$$

$$y = C e^{\int a(x)\,dx}$$

1) $$\underline{y' + \frac{1}{x} y = x^2}$$

$$y' + \frac{1}{x} y = 0$$

$$\frac{dy}{dx} = -\frac{1}{x} y$$

$$\int \frac{dy}{y} = \int -\frac{1}{x}\,dx$$

$$y = C e^{-\ln x} = C e^{\ln \frac{1}{x}}$$

$$\boxed{y_0 = C \frac{1}{x} = \frac{C}{x}}$$

$$\begin{cases} y = \dfrac{C(x)}{x} \\ y' = \dfrac{C' \cdot x - C \cdot 1}{x^2} \end{cases}$$

$$\frac{C'}{x} - \underbrace{\frac{C}{x^2} + \frac{1}{x}\frac{C}{x}} = x^2$$

$$\frac{C'}{x} = x^2 \longrightarrow C'(x) = x^3$$

$$\boxed{C(x) = \int x^3\,dx = \frac{x^4}{4} + C}$$

$$y = \frac{\frac{x^4}{4} + C}{x}$$

$$\boxed{y = \frac{C}{x} + \frac{x^3}{4}}$$

2) $$(x+1)y' - 2y = (x+1)^4 \qquad / : (x+1)$$

$$y' - \frac{2}{x+1} y = (x+1)^3$$

$$y' - \frac{2}{x+1} y = 0$$

$$\frac{dy}{dx} = \frac{2}{x+1} y$$

$$\int \frac{dy}{y} = \int \frac{2}{x+1}\,dx$$

$$y = C e^{2\ln(x+1)} = C e^{\ln (x+1)^2}$$

$$\boxed{y_0 = C(x+1)^2}$$

$$y = C(x) \cdot (x+1)^2$$

$$y' = C' \cdot (x+1)^2 + C \cdot 2(x+1) \cdot 1$$

$$C'(x+1)^2 + 2C(x+1) - \frac{2}{x+1} C(x+1)^2 = (x+1)^3$$

$$C'(x+1)^2 = (x+1)^3 \qquad / : (x+1)^2$$

$$C'(x) = x + 1$$

$$\boxed{C(x) = \int (x+1)\,dx = \frac{x^2}{2} + x + C}$$

$$y = \left(\frac{x^2}{2} + x + C\right) \cdot (x+1)^2$$

$$\boxed{y = C(x+1)^2 + \left(\frac{x^2}{2} + x\right)(x+1)^2}$$

$$\boxed{y'' + py' + qy = 0}$$

$\lambda^2 + p\lambda + q = 0$ charakteristická rovnice

1) $\lambda_1 \neq \lambda_2$ $\quad y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

2) $\lambda_{1,2} = \lambda$ $\quad y = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$

3) $\lambda_{1,2} = \alpha \pm \beta i$ $\quad y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$

3) $y'' - y' - 6y = 0$

$\lambda^2 - \lambda - 6 = 0$

$(\lambda - 3)(\lambda + 2) = 0$

$\lambda_1 = 3, \ \lambda_2 = -2$

$y = C_1 e^{3x} + C_2 e^{-2x}$

4) $y'' + 3y' = 0$

$\lambda^2 + 3\lambda = 0$

$\lambda(\lambda + 3) = 0$

$\lambda_1 = 0, \ \lambda_2 = -3$

$y = C_1 e^{0x} + C_2 e^{-3x}$

$y = C_1 + C_2 e^{-3x}$

5) $y'' - 4y = 0$

$\lambda^2 - 4 = 0 \quad (\lambda - 2)(\lambda + 2) = 0$

$\lambda_{1,2} = \pm 2$

$y = C_1 e^{2x} + C_2 e^{-2x}$

6) $y'' + 4y' + 4y = 0$

$\lambda^2 + 4\lambda + 4 = 0$ $\quad (a+b)^2$ $\quad (\lambda + 2)^2 = 0$

$D = 4^2 - 4 \cdot 4 \cdot 1 = 0$

$\lambda_{1,2} = \dfrac{-4 \pm \sqrt{0}}{2} = -2$

$$\boxed{y = C_1 e^{-2x} + C_2 x e^{-2x}}$$

7) $y'' - 2y' + 2y = 0$

$\lambda^2 - 2\lambda + 2 = 0$

$D = (-2)^2 - 4 \cdot 2 = 4 - 8 = -4$

$\lambda_{1,2} = \dfrac{2 \pm \sqrt{-4}}{2} = \dfrac{2 \pm i\sqrt{4}}{2}$

$\lambda_{1,2} = \dfrac{2 \pm 2i}{2} = 1 \pm i$ $\quad \alpha = 1, \ \beta = 1$

$y = e^{x}(C_1 \cos x + C_2 \sin x)$

8) $y'' + 2y' - 3y = 0$, $y(0) = -1$, $y'(0) = 7$

$\lambda^2 + 2\lambda - 3 = 0$

$(\lambda+3)(\lambda-1) = 0$

$\lambda_1 = -3$, $\lambda_2 = 1$

$\boxed{y = C_1 e^{-3x} + C_2 e^{x}}$

$-1 = C_1 e^{-3\cdot 0} + C_2 e^{0}$

$-1 = C_1 + C_2$

$C_2 = -C_1 - 1$

$y' = C_1(-3)e^{-3x} + C_2 e^{x}$

$7 = C_1 \cdot (-3) \cdot 1 + C_2 \cdot 1$

$7 = -3C_1 + C_2$

$7 = -3C_1 - C_1 - 1$

$7 = -4C_1 - 1 \rightarrow 8 = -4C_1 \rightarrow C_1 = -2$

$C_2 = -(-2) - 1 = 1$

$\boxed{y = -2e^{-3x} + e^{x}}$

9) $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = 6$

$\lambda^2 + 4 = 0$ $\qquad \lambda^2 = -4$

$D = 0^2 - 4 \cdot 4 = -16$

$\lambda_{1,2} = \frac{0 \pm \sqrt{-16}}{2} = \frac{\pm i\sqrt{16}}{2} = \frac{\pm 4i}{2} = \pm 2i$

$\alpha = 0$

$\beta = 2$

$y = e^{0x}(C_1 \cos 2x + C_2 \sin 2x)$

$\boxed{y = C_1 \cos 2x + C_2 \sin 2x}$

$0 = C_1 \overset{=1}{\cos 2\cdot 0} + C_2 \underset{=0}{\sin 2\cdot 0}$

$0 = C_1$

$y' = C_1(-\sin 2x)\cdot 2 + C_2 \cos 2x \cdot 2$

$y' = -2C_1 \sin 2x + 2C_2 \cos 2x$

$6 = -2C_1 \overset{=0}{\sin 0} + 2C_2 \overset{=1}{\cos 0}$

$6 = 2C_2 \rightarrow C_2 = 3$

$\boxed{y = 3\sin 2x}$

$y_1 = e^{2x}$, $y_2 = e^{-x}$? $y = C_1 e^{2x} + C_2 e^{-x}$

$\lambda_1 = 2$, $\lambda_2 = -1$

$(\lambda - 2)(\lambda + 1) = 0$

$\lambda^2 - \lambda - 2 = 0$

$\boxed{y'' - y' - 2y = 0}$

$y_1 = e^x$, $y_2 = e^{3x}$

$\lambda_1 = 1$, $\lambda_2 = 3$

$(\lambda - 1)(\lambda - 3) = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$y'' - 4y' + 3y = 0$

$y_1 = xe^{2x}$ $y_2 = e^{2x}$

$\lambda_{1,2} = 2$

$(\lambda - 2)(\lambda - 2) = (\lambda - 2)^2 = 0$

$\lambda^2 - 4\lambda + 4 = 0$

$y'' - 4y' + 4y = 0$

$y_1 = \sin x$ $y_2 = \cos x$

$\lambda_{1,2} = \pm i$ $\sqrt{-1} = i$

$(\lambda - i)(\lambda + i) = 0$ $i^2 = -1$

$\lambda^2 - i\lambda + i\lambda - i^2 = 0$

$\lambda^2 + 1 = 0$

$\boxed{y'' + y = 0}$

$y_1 = e^x \cos 2x$ $y_2 = e^x \sin 2x$

$\lambda_{1,2} = 1 \pm 2i$

$[\lambda - (1 + 2i)] \cdot [\lambda - (1 - 2i)] = 0$ $(a - b)(a + b)$

$[(\lambda - 1) - 2i] \cdot [(\lambda - 1) + 2i] = 0$ $a^2 - b^2$

$(\lambda - 1)^2 - (2i)^2 = 0$

$\lambda^2 - 2\lambda + 1 - (-4) = 0$ $\lambda^2 - 2\lambda + 5 = 0$

$\boxed{y'' - 2y' + 5y = 0}$

$y'' + py' + qy = f(x)$