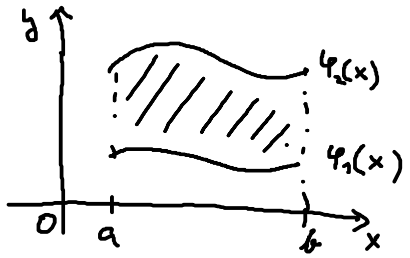
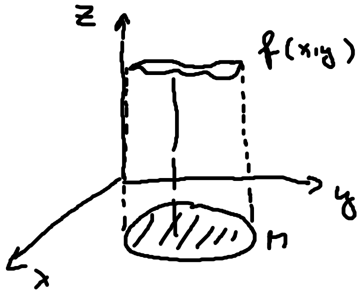
$\iint\limits_{M} f(x,y)\,dx\,dy$, M: $a \le x \le b$, $\varphi_1(x) \le y \le \varphi_2(x)$



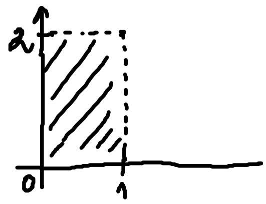
$$\int_a^b \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right) dx$$



$$V = \iint\limits_M f(x,y)\,dx\,dy$$

$$|M| = \iint\limits_M 1\,dx\,dy$$

Př. 1) $\iint\limits_M (x-y)\,dx\,dy$, M:



$0 \le x \le 1$
$0 \le y \le 2$

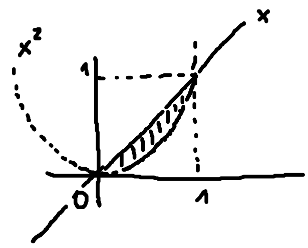
$$\iint\limits_M (x-y)\,dx\,dy = \int_0^1 \left(\int_0^2 (x-y)\,dy \right) dx = \int_0^1 \left[xy - \frac{y^2}{2} \right]_0^2 dx = \int_0^1 (2x-2)\,dx =$$

$$= \left[x^2 - 2x \right]_0^1 = 1 - 2 = -1$$

$$\iint\limits_M (x-y)\,dx\,dy = \int_0^2 \left(\int_0^1 (x-y)\,dx \right) dy = \int_0^2 \left[\frac{x^2}{2} - yx \right]_0^1 dy = \int_0^2 \left(\frac{1}{2} - y \right) dy =$$

$$= \left[\frac{1}{2}y - \frac{y^2}{2} \right]_0^2 = \frac{1}{2}\cdot 2 - 2 = -1$$

Př. 2 $\iint\limits_M xy^2\,dx\,dy$, M: $y = x$, $y = x^2$



$0 \le x \le 1$
$x^2 \le y \le x$

$$\int_0^1 \left(\int_{x^2}^{x} xy^2\,dy \right) dx = \int_0^1 \left[x\frac{y^3}{3} \right]_{x^2}^{x} dx = \int_0^1 \left(x\frac{x^3}{3} - x\frac{x^6}{3} \right) dx =$$

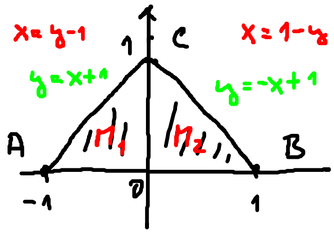
$$= \frac{1}{3}\int_0^1 (x^4 - x^7)\,dx = \frac{1}{3}\left[\frac{x^5}{5} - \frac{x^8}{8} \right]_0^1 = \frac{1}{3}\left(\frac{1}{5} - \frac{1}{8} \right) = \frac{1}{40}$$

$|M| = ?$

$$|M| = \iint\limits_M 1\,dx\,dy = \int_0^1 \left(\int_{x^2}^{x} 1\,dy \right) dx = \int_0^1 \left[y \right]_{x^2}^{x} dx = \int_0^1 (x - x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

3) $\iint_M x\,dx\,dy$, $M:\ \triangle ABC$, $A=[-1,0]$, $B\,[1,0]$, $C=[0,1]$



$M_1:\ -1 \le x \le 0$, $\quad 0 \le y \le x+1$

$M_2:\ 0 \le x \le 1$, $\quad 0 \le y \le -x+1$

$M:\ 0 \le y \le 1$, $\quad y-1 \le x \le 1-y$

$$\iint_M x\,dx\,dy = \iint_{M_1} x\,dx\,dy + \iint_{M_2} x\,dx\,dy$$

$$\iint_{M_1} x\,dx\,dy = \int_{-1}^{0}\left(\int_0^{x+1} x\,dy\right)dx = \int_{-1}^{0}\left[xy\right]_0^{x+1}dx = \int_{-1}^{0} x(x+1)\,dx = \int_{-1}^{0}(x^2+x)\,dx$$

$$= \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{-1}^{0} = 0-\left(\frac{(-1)^3}{3}+\frac{(-1)^2}{2}\right) = -\left(-\frac{1}{3}+\frac{1}{2}\right) = -\frac{1}{6}$$

$$\iint_{M_2} x\,dx\,dy = \int_0^1\left(\int_0^{1-x} x\,dy\right)dx = \int_0^1\left[xy\right]_0^{1-x}dx = \int_0^1 x(1-x)\,dx = \int_0^1(x-x^2)\,dx =$$

$$= \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 = \frac{1}{2}-\frac{1}{3} = \frac{1}{6}$$

$$\iint_M x\,dx\,dy = -\frac{1}{6}+\frac{1}{6} = 0$$

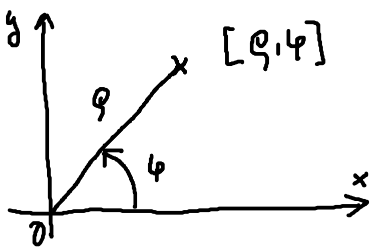
Dú

$\iint_M y\,dx\,dy$

$(1-y)^2 = (y-1)^2$

$$\iint_M x\,dx\,dy = \int_0^1\left(\int_{y-1}^{1-y} x\,dx\right)dy = \int_0^1\left[\frac{x^2}{2}\right]_{y-1}^{1-y}dy = \int_0^1\left(\frac{(1-y)^2}{2}-\frac{(y-1)^2}{2}\right)dy = \int_0^1 0\,dy$$

$$= 0$$



$$x = \rho\cos\varphi$$

$$y = \rho\sin\varphi$$

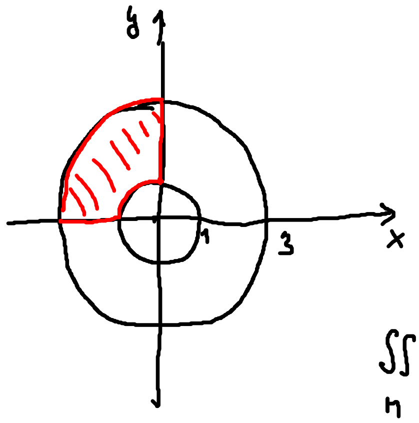
$$x^2+y^2 = \rho^2$$

$$x^2+y^2 = \rho^2\cos^2\varphi + \rho^2\sin^2\varphi = \rho^2(\cos^2\varphi+\sin^2\varphi) = \rho^2$$

$$\iint_M f(x,y)\,dx\,dy = \int_{\varphi_1}^{\varphi_2}\left(\int_{\rho_1(\varphi)}^{\rho_2(\varphi)} f(\rho\cos\varphi,\rho\sin\varphi)\,\rho\,d\rho\right)d\varphi$$

4) $\iint\limits_M (x^2+y^2)\,dx\,dy$, $M:$ $\underline{1 \le x^2+y^2 \le 9}$, $y \ge 0$, $x \le 0$

$x^2+y^2 \ge 1 \;\wedge\; x^2+y^2 \le 9$



$1 \le \varrho \le 3$

$\frac{\pi}{2} \le \varphi \le \pi$

$\frac{9\pi}{4} - \frac{\pi}{4} = \underline{\underline{2\pi}}$

$$\iint\limits_M (x^2+y^2)\,dx\,dy = \int_{\frac{\pi}{2}}^{\pi} \left(\int_1^3 \varrho^2 \cdot \varrho \, d\varrho \right) d\varphi =$$

$$= \int_{\frac{\pi}{2}}^{\pi} \left[\frac{\varrho^4}{4} \right]_1^3 d\varphi = \int_{\frac{\pi}{2}}^{\pi} \left(\frac{81}{4} - \frac{1}{4} \right) d\varphi = \int_{\frac{\pi}{2}}^{\pi} 20 \, d\varphi =$$

$$= \left[20\varphi \right]_{\pi/2}^{\pi} = 20\pi - 20 \cdot \frac{\pi}{2} = \underline{\underline{10\pi}}$$

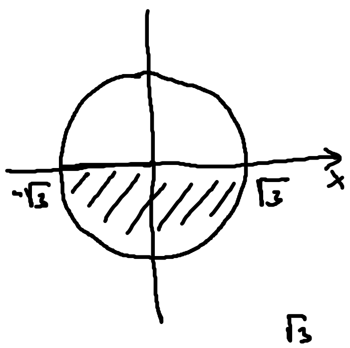
$$\int_{\frac{\pi}{2}}^{\pi} \left(\int_1^3 \varrho^3 d\varrho \right) d\varphi = \int_{\frac{\pi}{2}}^{\pi} d\varphi \cdot \int_1^3 \varrho^3 d\varrho = \left[\varphi \right]_{\frac{\pi}{2}}^{\pi} \cdot \left[\frac{\varrho^4}{4} \right]_1^3 = \left(\pi - \frac{\pi}{2}\right) \cdot \left(\frac{81}{4} - \frac{1}{4} \right)$$

$$= \underline{\underline{10\pi}}$$

$$|M| = \iint\limits_M 1 \, dx\,dy = \int_{\frac{\pi}{2}}^{\pi} \left(\int_1^3 \varrho \, d\varrho \right) d\varphi = \int_{\frac{\pi}{2}}^{\pi} d\varphi \cdot \int_1^3 \varrho \, d\varrho = \left[\varphi \right]_{\frac{\pi}{2}}^{\pi} \cdot \left[\frac{\varrho^2}{2} \right]_1^3 =$$

$$= \left(\pi - \frac{\pi}{2} \right) \cdot \left(\frac{9}{2} - \frac{1}{2} \right) = \frac{\pi}{2} \cdot 4 = \underline{\underline{2\pi}} \quad j^2$$

5) $\iint\limits_M (x+y)\,dx\,dy$, $M:$ $x^2+y^2 \le 3$, $y \le 0$



$0 \le \varrho \le \sqrt{3}$

$\pi \le \varphi \le 2\pi$

$$\int_{\pi}^{2\pi} \left(\int_0^{\sqrt{3}} (\varrho\cos\varphi + \varrho\sin\varphi)\varrho \, d\varrho \right) d\varphi =$$

$$= \int_{\pi}^{2\pi} \left(\int_0^{\sqrt{3}} \varrho^2 (\cos\varphi + \sin\varphi) \, d\varrho \right) d\varphi =$$

$$\int_{\pi}^{2\pi} (\cos\varphi + \sin\varphi)\, d\varphi \cdot \int_0^{\sqrt{3}} \varrho^2 d\varrho = \left[\sin\varphi - \cos\varphi \right]_{\pi}^{2\pi} \cdot \left[\frac{\varrho^3}{3} \right]_0^{\sqrt{3}} =$$

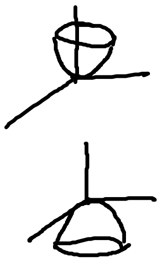
$$= \left[\left(\overset{0}{\sin 2\pi} - \overset{1}{\cos 2\pi} \right) - \left(\overset{0}{\sin\pi} - \overset{-1}{\cos\pi} \right) \right] \cdot \frac{3\sqrt{3}}{3} = -2\sqrt{3}$$

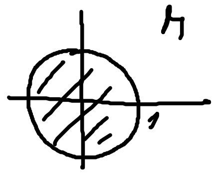
6) Vyp. objem tělesa ohraničeného paraboloidem $f(x,y) = 4 - x^2 - y^2$

na množině $M: \; x^2 + y^2 \le 1$.

$z = x^2 + y^2$

$z = -(x^2 + y^2)$





$z = 4 - (x^2 + y^2)$

$M: \; 0 \le \varrho \le 1$

$0 \le \varphi \le 2\pi$

$0 = 4 - (x^2 + y^2)$

$x^2 + y^2 = 4$

$$V = \iint\limits_M (4 - x^2 - y^2)\, dx\, dy = \int_0^{2\pi} \left(\int_0^1 (4 - \varrho^2)\, \varrho \, d\varrho \right) d\varphi =$$

$$= \int_0^{2\pi} d\varphi \cdot \int_0^1 (4\varrho - \varrho^3)\, d\varrho =$$