$z = f(x,y)$ $\quad z_x, z_y$

$z_{xx}, \underline{z_{xy} = z_{yx}}, z_{yy}$

1) Vyp. druhé der.

a) $z = \ln \frac{x}{y-x}$

$$z_x = \frac{1}{\frac{x}{y-x}} \cdot \frac{1\cdot(y-x) - x\cdot(-1)}{(y-x)^2} = \frac{y-x}{x} \cdot \frac{y}{(y-x)^2} = \frac{y}{x(y-x)}$$

$$z_y = \frac{y-x}{x} \cdot \frac{0 - x\cdot 1}{(y-x)^2} = -\frac{1}{y-x}$$

$$z_{xx} = \left(\frac{y}{x(y-x)}\right)'_x = \frac{0 - y\cdot(y-2x)}{x^2(y-x)^2} = \frac{+2xy - y^2}{x^2(y-x)^2}$$

$xy - x^2$

$$z_{xy} = \left(\frac{y}{x(y-x)}\right)'_y = \frac{1\cdot x(y-x) - yx}{x^2(y-x)^2} = \frac{-x^2}{x^2(y-x)^2} = -\frac{1}{(y-x)^2}$$

$$z_{yx} = \left(-\frac{1}{y-x}\right)'_x = -\frac{0 - 1\cdot(-1)}{(y-x)^2} = -\frac{1}{(y-x)^2}$$

Schwarzova věta

$$z_{yy} = \left(-\frac{1}{y-x}\right)'_y = -\frac{0 - 1\cdot 1}{(y-x)^2} = \frac{1}{(y-x)^2}$$

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b) $z = \operatorname{arctg} \frac{y}{x}$ $\qquad \frac{y}{x} = y \cdot \frac{1}{x} = y \cdot x^{-1}$

$$z_x = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot y \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{\frac{x^2+y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = \frac{x^2}{x^2+y^2}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$z_y = \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$z_{xy} = \left(-\frac{y}{x^2+y^2}\right)'_y = -\frac{1\cdot(x^2+y^2) - y\cdot 2y}{(x^2+y^2)^2} = \frac{-(x^2-y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$z_{yx} = \left(\frac{x}{x^2+y^2}\right)'_x = \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

2) Vypočtěte totální diferenciál

a) $z = f(x,y) = 3xy^2 - xy$ v bodě $[2,1]$, $dx = 0{,}1$; $dy = -0{,}1$

$$df(x_0,y_0) = f_x(x_0,y_0)\,dx + f_y(x_0,y_0)\,dy$$

$f_x = 3y^2 - y$ $\quad f_x(2,1) = 2$

$f_y = 6xy - x$ $\quad f_y(2,1) = 10$

$$df(2,1) = 2 \cdot 0{,}1 + 10 \cdot (-0{,}1) = -0{,}8$$

$f(x,y) = 3xy^2 - xy$ $\quad f(2,1) = 3 \cdot 2 \cdot 1^2 - 2 \cdot 1 = 4$

$$f(2{,}1;\, 0{,}9) = 3 \cdot 2{,}1 \cdot 0{,}9^2 - 2{,}1 \cdot 0{,}9 = 3{,}21$$

b) $z = f(x,y) = \dfrac{y}{x^2+y^2}$, v bodě $[1,1]$

$$z_x = \frac{0 - y \cdot 2x}{(x^2+y^2)^2} = \frac{-2}{2^2} = -\frac{1}{2}$$

$$z_y = \frac{1(x^2+y^2) - y \cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} = 0$$

$$df(1,1) = -\frac{1}{2}\,dx + 0\,dy = -\frac{1}{2}\,dx$$

$$f(1,1) = \frac{1}{1+1} = \frac{1}{2}$$

pro $dx = -0{,}1$, $dy = 0{,}1$

$$f(0{,}9;\, 1{,}1) \doteq \frac{1}{2} + df(1,1) = \frac{1}{2} + \left(-\frac{1}{2}\right) \cdot (-0{,}1) = \frac{1}{2} + 0{,}05 = 0{,}55$$

c) pomocí dif. vypočtěte přibližnou hodnotu

$1{,}02^5 \cdot 0{,}99^{20} \doteq ?$ $\qquad f(x,y) = x^5 y^{20}$, $[1,1]$, $dx = 0{,}02$, $dy = -0{,}01$

$$1{,}02^5 \cdot 0{,}99^{20} \doteq 1 + df(1,1) = 1 - 0{,}1 = 0{,}9$$

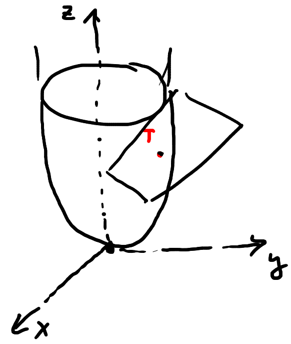
$f_x = 5x^4 y^{20}$ $\quad f_x(1,1) = 5$

$f_y = x^5\, 20 \cdot x^{19}$ $\quad f_y(1,1) = 20$

$$df(1,1) = 5 \cdot 0{,}02 + 20(-0{,}01) = 0{,}1 - 0{,}2 = -0{,}1$$

3) Tečná rovina

$$z = x^2 + y^2$$



$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

a)

$z = f(x,y) = x^3 + x^2y + 2y^2$ $\quad [1,1,?]$ $\quad f(1,1) = 4$

$[1,1,4]$

$z_x = 3x^2 + 2xy \qquad f_x(1,1) = 5$

$z_y = x^2 + 4y \qquad f_y(1,1) = 5$

$$z = 4 + 5(x-1) + 5(y-1)$$

$$z = 4 + 5x - 5 + 5y - 5$$

$$\boxed{5x + 5y - z - 6 = 0}$$

b) $f(x,y) = x^2 + y^2 \qquad [1,2,5]$

$f_x = 2x \qquad f_x(1,2) = 2$

$f_y = 2y \qquad f_y(1,2) = 4$

$$z = 5 + 2(x-1) + 4(y-2)$$

$z = f(x,y)$, Lokální extrémy

$z_x = 0$
$z_y = 0$ $\longrightarrow$ stacionární body

$$D(x_0,y_0) = z_{xx}(x_0,y_0) \cdot z_{yy}(x_0,y_0) - \left(z_{xy}(x_0,y_0)\right)^2$$

D:
- > 0 je extrém: $z_{xx} > 0$ – min, $z_{xx} < 0$ – max
- < 0 není extrém
- $= 0$?

$$\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = z_{xx} z_{yy} - (z_{xy})^2$$

Vypočítejte lok. extrémy fce

$$f(x,y) = 2x^3 - 3x^2 + y^2 - 2$$

$f_x = 6x^2 - 6x$

$f_y = 2y$

$6x^2 - 6x = 0$
$2y = 0$

$x^2 - x = 0$
$y = 0$

$x(x-1) = 0$

$x_1 = 0$ $x_2 = 1$

$[0,0]$, $[1,0]$

$f_{xx} = 12x - 6$

$f_{xy} = 0$

$f_{yx} = 0$

$f_{yy} = 2$

$D(0,0) = -6 \cdot 2 - 0^2 = -12 < 0$ není extrém

$D(1,0) = 6 \cdot 2 - 0^2 = 12 > 0$ je extrém

$6 > 0$

v bodě $[1,0]$ je lokální minimum

$f(1,0) = -3$