6. cvičení z lineární algebry II

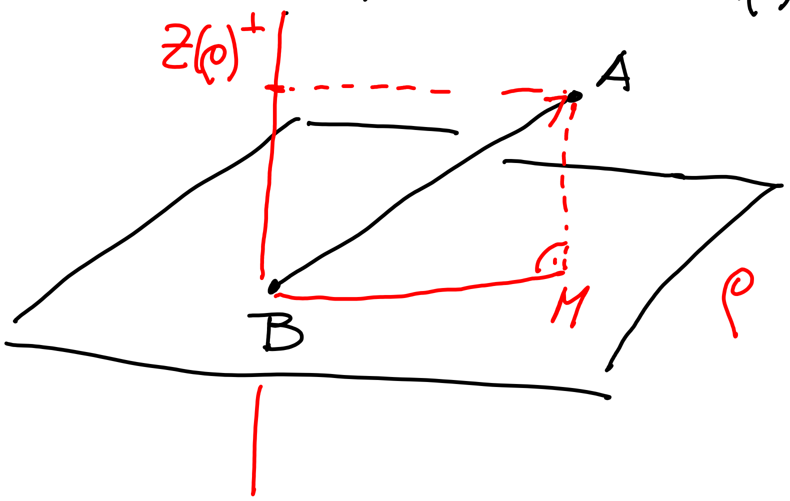
Příklad. 1. V $\mathbb{R}^4$ určete vzdálenost bodu $A = [4, 1, -4, -5]$ od roviny

$$\rho: \ [3, -2, 1, 5] + t(2, 3, -2, -2) + s(4, 1, 3, 2).$$

Současně najděte bod $M \in \rho$ takový, že $\|M - A\| = \operatorname{dist}(A, \rho)$.

$B = [3, -2, 1, 5] \qquad u_1 = (2, 3, -2, -2) \qquad u_2 = (4, 1, 3, 2)$

Vzdálenost $\operatorname{dist}(A, \rho) = \| P_{Z(\rho)^\perp}(A - B) \|$



$A - B = (1, 3, -5, -10)$ Spočítáme kolmou projekci do $Z(\rho) = [u_1, u_2]$.

$$P_{Z(\rho)}(A - B) = a u_1 + b u_2$$

$$(A - B) - P_{Z(\rho)}(A - B) \perp \underset{u_1,\ u_2}{Z(\rho)}$$

$$\langle (A - B) - a u_1 - b u_2, u_1 \rangle = 0$$
$$\langle (A - B) - a u_1 - b u_2, u_2 \rangle = 0$$

$$a \langle u_1, u_1 \rangle + b \langle u_2, u_1 \rangle = \langle A - B, u_1 \rangle$$
$$a \langle u_1, u_2 \rangle + b \langle u_2, u_2 \rangle = \langle A - B, u_2 \rangle$$

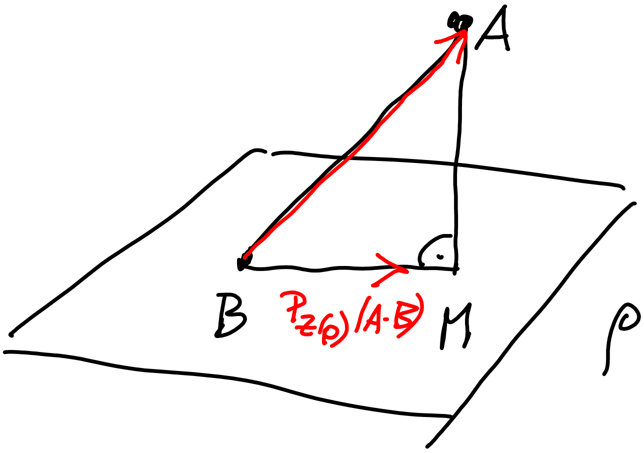
$$\begin{aligned} 21a + b &= 41 \\ a + 30b &= -28 \end{aligned}$$

$$\left(\begin{array}{cc|c} 21 & 1 & 41 \\ 1 & 30 & -28 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 30 & -28 \\ 0 & -629 & 629 \end{array}\right) \qquad \begin{array}{l} b = -1 \\ a = 2 \end{array}$$

$$P_{Z(\rho)}(A-B) = 2\cdot u_1 - u_2 = (0,5,-7,-6).$$

$$P_{Z(\rho)^\perp}(A-B) = (A-B) - P_{Z(\rho)} = (1,3,-5,-10) - (0,5,-7,-6) = (1,-2,2,-4)$$

$$\underline{\underline{\operatorname{dist}(A,\rho)}} = \|(1,-2,2,4)\| = \sqrt{1+4+4+16} = \underline{\underline{5}}$$



$$\underline{\underline{M}} = B + P_{Z(\rho)}(A-B) =$$

$$= [3,-2,1,5] + (0,5,-7,-6) = \underline{\underline{[3,3,-6,-1]}}$$

Příklad. 2. V $\mathbb{R}^4$ určete vzdálenost přímky p od roviny ρ

$$p: \ [5,4,4,5] + r(0,0,1,-4), \qquad \rho: \ [4,1,1,0] + s(1,-1,0,0) + t(2,0,-1,0)$$

a body $M \in p$ a $N \in \rho$, v nichž se tato vzdálenost realizuje, tj. $\|M - N\| = \operatorname{dist}(p,\rho)$.

$p: \ A + r u \qquad A = [5,4,4,5] \quad u = (0,0,1,-4)$

$\rho: \ B + s v_1 + t v_2 \qquad B = [4,1,1,0], \ v_1 = (1,-1,0,0)$

$v_2 = (2,0,-1,0)$

$A - B = (1,3,3,5)$

Pokud chceme vzdálenost, je nejlepší spočítat $P_{(Z(p)+Z(\rho))^\perp}(A-B)$

a její velikost je hledaná vzdálenost.

$(Z(p)+Z(\rho))^\perp = \{x \in \mathbb{R}^4, \ x \perp u, v_1, v_2\}$

$\langle x, u\rangle = 0$

$\langle x, v_1\rangle = 0$

$\langle x, v_2\rangle = 0$

$$\begin{pmatrix} 0 & 0 & 1 & -4 \\ 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

$x = a \cdot (2,2,4,1)$

kolmá projekce do $(Z(p)+Z(\rho))^\perp = [(2,2,4,1)]$

je $P(A-B) = a \cdot (2,2,4,1)$

$A - B - a(2,2,4,1) \perp (2,2,4,1)$

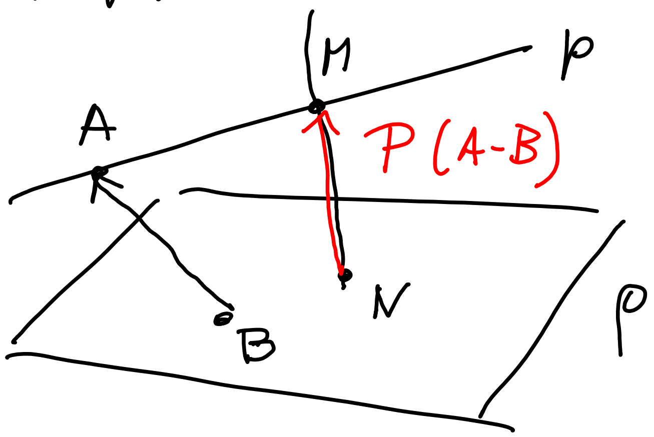
$$a = \frac{\langle A-B, (2,2,4,1)\rangle}{\langle (2,2,4,1), (2,2,4,1)\rangle} =$$

$$= \frac{\langle (1,3,3,5),(2,2,4,1)\rangle}{25} = \frac{25}{25}$$

$$a = 1$$

$$P(A-B) = \underline{(2,2,4,1)}$$

$$\mathrm{dist}(p,\rho) = \|(2,2,4,1)\| = \underline{\underline{5}}$$



$$A-B-P(A-B) = (1,3,3,5) - (2,2,4,1) = (-1,1,-1,4)$$

$$M = A + \underline{r}u$$
$$N = B + \underline{s}v_1 + \underline{t}v_2$$

$$M = N + P(A-B)$$

$$A + ru = B + sv_1 + tv_2 + P(A-B)$$

$$A - B - P(A-B) = sv_1 + tv_2 - ru$$

$$\underline{s}v_1 + \underline{t}v_2 - \underline{r}u = (A-B) - P(A-B)$$

4 rovnice o 3 neznámých

$$\begin{array}{cccc} s & t & r & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 4 & 4 \end{array}\right) \sim$$

$$r = 1$$
$$t = 0$$
$$s = -1$$

$$M = A + u = [5,4,5,1]$$
$$N = B - v_1 = [3,2,1,0]$$

Jiný postup řešení

Hledáme přímo body M a N, kde se vzdálenost realizuje.

$$M - N \perp Z(\rho) + Z(p)$$

$$\perp u, v_1, v_2$$

$$\langle A + \underline{r}u - B - \underline{s}v_1 - \underline{t}v_1, u \rangle = 0 \qquad (3)$$

$$\langle \text{——}''\text{——}, v_1 \rangle = 0 \qquad (1)$$

$$\langle \text{——}''\text{——}, v_2 \rangle = 0 \qquad (2)$$

Po úpravě

$$s\langle v_1, v_1\rangle + t\langle v_2, v_1\rangle - r\langle u, v_1\rangle = \langle A-B, v_1\rangle$$

$$s\langle v_1, v_2\rangle + t\langle v_2, v_2\rangle - r\langle u, v_2\rangle = \langle A-B, v_2\rangle$$

$$s\langle v_1, u\rangle + t\langle v_2, u\rangle - r\langle u, u\rangle = \langle A-B, u\rangle$$

Číselně

$$\left(\begin{array}{ccc|c} 2 & 2 & 0 & -2 \\ 2 & 5 & 1 & -1 \\ 0 & -1 & -17 & -17 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & -1 & -17 & -17 \\ 0 & 3 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & -1 & -17 & -17 \\ 0 & 0 & -50 & -50 \end{array}\right) \qquad \begin{array}{l} r = 1 \\ t = 0 \\ s = -1 \end{array}$$

$$M = A + 1.u = [5,4,5,1]$$

$$N = B + (-1)v_1 = [3,2,1,0]$$

$$\underline{\underline{\operatorname{dist}(p,\rho)}} = \|M-N\| = \|(2,2,4,1)\| = \underline{\underline{5}}$$

Příklad. 3. V $\mathbb{R}^4$ určete vzdálenost rovin σ a τ a body, v nichž se realizuje.

$$\sigma : [4,5,3,2] + s(1,2,2,2) + t(2,0,2,1),$$
$$\tau : [1,-2,1,-3] + p(2,-2,1,2) + q(1,-2,0,-1).$$

$\sigma : A + su_1 + tu_2 \qquad A = [4,5,3,2], \; u_1 = (1,2,2,2)$

$u_2 = (2,0,2,1)$

$\tau : B + pv_1 + qv_2 \qquad B = [1,-2,1,-3] \quad v_1 = (2,-2,1,2)$

$v_2 = (1,-2,0,-1)$

Použijeme 1. metodu z předch. příkladu

Počítáme kolmou projekci vektoru

$A - B = (3,7,2,5)$ do $\left(Z(\sigma) + Z(\tau)\right)^{\perp} =$

$= \{ x \in \mathbb{R}^4, \; x \perp u_1, u_2, v_1, v_2 \}$

Soustava 4 rovnic a 4 neznámých

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 0 & 2 & 1 \\ 2 & -2 & 1 & 2 \\ 1 & -2 & 0 & -1 \end{pmatrix} \sim \dots \sim \begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x = a(2,1,-2,0)$

Kolmá projekce $A - B$ do $[(2,1,-2,0)]$

je $\quad P(A-B) = a(2,1,-2,0)$

$A - B - a(2,1,-2,0) \perp (2,1,-2,0)$

$$a = \frac{\langle A-B, (2,1,-2,0)\rangle}{\|(2,1,-2,0)\|^2} = \frac{9}{9} = 1$$

$(3,7,2,5) - (2,1,-2,0)$

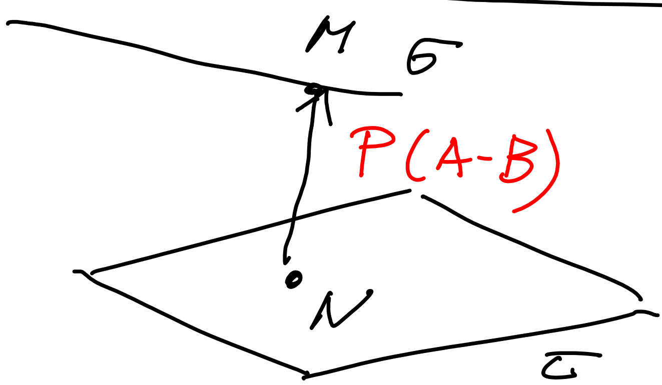
$P(A-B) = (2,1,-2,0)$

$\underline{\text{dist}(\sigma,\tau)} = \|(2,1,-2,0)\| = \underline{3}$

Nyní hledáme $M \in \sigma$, $N \in \tau$

$$\|M-N\| = 3$$

$$\boxed{M = N + P(A-B)}$$



$$M = A + \underline{s}u_1 + \underline{t}u_2 \qquad N = B + \underline{p}v_1 + \underline{q}v_2$$

$$A + su_1 + tu_2 = B + pv_1 + qv_2 + P(A-B)$$

$$(A-B) - P(A-B) = -\underline{s}u_1 - \underline{t}u_2 + \underline{p}v_1 + \underline{q}v_2$$

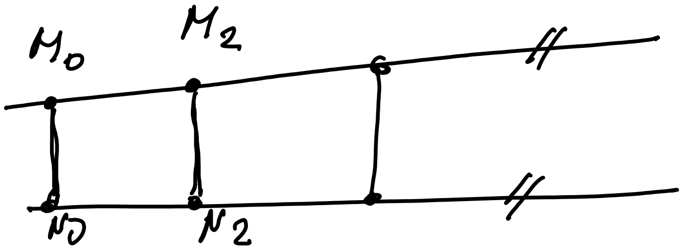
$$\left(\begin{array}{cccc|c} -1 & -2 & 2 & 1 & 1 \\ -2 & 0 & -2 & -2 & 6 \\ -2 & -2 & 1 & 0 & 4 \\ -2 & -1 & 2 & -1 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & -2 & -1 & -1 \\ 0 & 4 & -6 & -4 & 4 \\ 0 & 2 & -3 & -2 & 2 \\ 0 & 3 & -2 & -3 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & -2 & -1 & -1 \\ 0 & 2 & -3 & -2 & 2 \\ 0 & 3 & -2 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & -2 & -1 & -1 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$(s, t, p, q) = (-3-b,\ 1+b,\ 0,\ b)$$

$$\begin{aligned} M_b &= [4,5,3,2] - (3+b)(1,2,2,2) \\ &\qquad + (1+b)(2,0,2,1) \\ &= [3,-1,-1,-3] + b(1,-2,0,-1) \end{aligned}$$

$$N_b = [1,-2,1,3] + b(1,-2,0,-1)$$



$M_b \in \sigma$

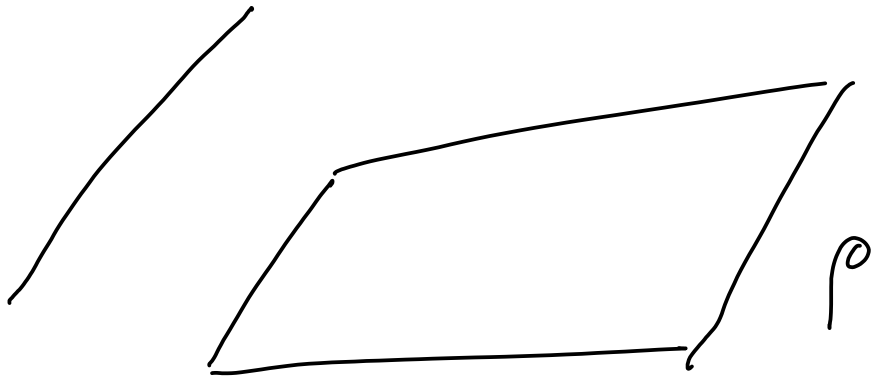
$N_b \in \tau$

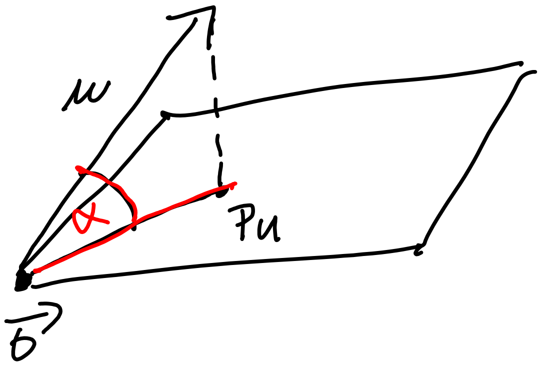
$$M_b - N_b = (2, 1, -2, 0)$$

$$\| M_b - N_b \| = 3$$

u

Příklad. 4. Určete odchylku přímky $p:\ [1,2,3,4]+t(-3,15,1,-5)$ od roviny
$$\rho:\ [0,0,0,0]+r(1,-5,-2,10)+s(1,8,-2,-16).$$





$$\cos\alpha = \frac{\|Pu\|}{\|u\|}$$

$u = (-3, 15, 1, -5)$

$Z(\rho) = [v_1, v_2]$ $\qquad v_1 = (1, -5, 2, 10)$

$\qquad v_2 = (1, 8, -2, -16)$

P kolmá projekce do $Z(\rho)$

$$Pu = a v_1 + b v_2$$

$$u - Pu \perp v_1, v_2$$

$$a\langle v_1, v_1\rangle + b\langle v_2, v_1\rangle = \langle u, v_1\rangle$$
$$a\langle v_1, v_2\rangle + b\langle v_2, v_2\rangle = \langle u, v_2\rangle$$

$$\left(\begin{array}{cc|c} 130 & -195 & -130 \\ -195 & 325 & 195 \end{array}\right)$$

Řešení uhádneme
$a = -1 \quad b = 0$

$$Pu = (-1) \cdot v_1 = (-1, 5, 2, -10)$$

$$\cos\alpha = \frac{\|Pu\|}{\|u\|} = \frac{\|(-1, 5, 2, 10)\|}{\|(-3, 15, 1, -5)\|} = \frac{\sqrt{130}}{\sqrt{260}}$$

$$= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \alpha = \frac{\pi}{4}$$

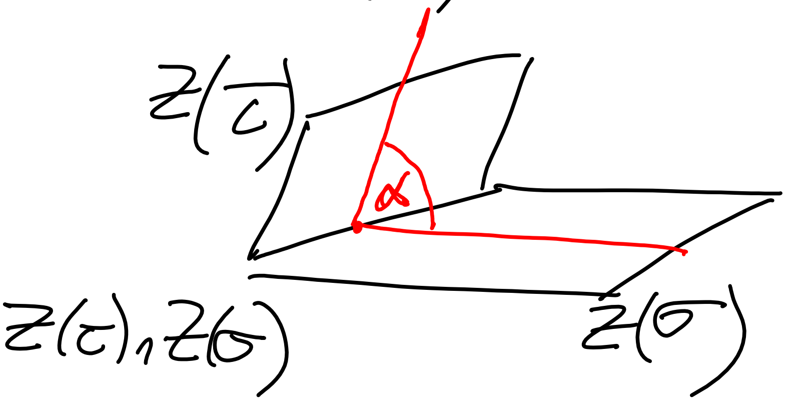
$$\alpha \in \left[0, \frac{\pi}{2}\right]$$

Odchylka roviny a přímky je $\frac{\pi}{4}$.

Příklad. 5. V $\mathbb{R}^4$ určete odchylku rovin τ a σ.

$$\sigma : [2,1,0,1] + s(1,1,1,1) + t(1,-1,1-1),$$
$$\tau : [1,0,1,1] + p(2,2,1,0) + q(1,-2,2,0).$$

$$\sphericalangle(\tau,\sigma) = \sphericalangle(Z(\tau), Z(\sigma))$$



$$Z(\sigma)\cap Z(\tau) = [(1,0,1,0)]$$

$$\sphericalangle(Z(\sigma), Z(\tau)) = \sphericalangle\big(Z(\sigma)\cap(Z(\sigma)\cap Z(\tau))^{\perp},\ Z(\tau)\cap(Z(\sigma)\cap Z(\tau))^{\perp}\big)$$

$$(Z(\sigma)\cap Z(\tau))^{\perp} = [(1,0,-1,0),(0,1,0,0),(0,0,0,1)]$$

$$P = Z(\sigma)\cap(Z(\sigma)\cap Z(\tau))^{\perp} =$$
$$= [(1,1,1,1),(1,-1,1,-1)] \cap [(1,0,-1,0),(0,1,0,0),(0,0,0,1)]$$
$$= [(0,1,0,1)]$$

$$Q = Z(\tau)\cap(Z(\sigma)\cap Z(\tau))^{\perp} =$$
$$= [(2,2,1,0),(1,-2,2,0)] \cap [(1,0,-1,0),(0,1,0,0),(0,0,0,1)]$$
$$= [(1,4,-1,0)]$$

$\measuredangle(P, Q) = \measuredangle(Z(\sigma), Z(\tau)) = \alpha \in [0, \frac{\pi}{2}]$

$\swarrow$ priamky generované vektory

$$\cos\alpha = \frac{|\langle (0,1,0,1), (1,4,-1,0)\rangle|}{\|(0,1,0,1)\| \cdot \|(1,4,-1,0)\|} =$$

$$= \frac{4}{\sqrt{2}\cdot\sqrt{18}} = \frac{4}{2\cdot 3} = \frac{2}{3}$$

Odchýlka je uhol $\alpha \in [0, \frac{\pi}{2}]$ taký, že

$$\cos\alpha = \frac{2}{3}.$$

Příklad. 6. V $\mathbb{R}^5$ spočítejte odchylku roviny ρ a nadroviny Γ.

$$\rho : s(1,-1,1,1,3) + t(1,-3,-3,-3,-9),$$
$$\Gamma : x_1 + 2x_2 - x_3 + 3x_4 + x_5 = 0.$$

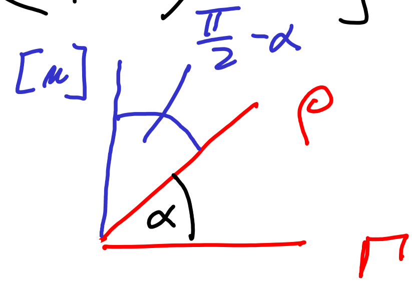
$\dim \rho = 2$
$\dim \Gamma = 4$

$\Gamma = Z(\Gamma)$ $\qquad Z(\Gamma)^\perp = [n]$

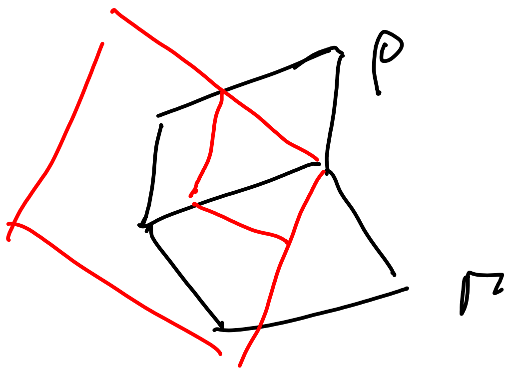
n normálový vektor

$\underline{n = (1,2,-1,3,1)}$

$Z(\Gamma) = \{x \in \mathbb{R}^5, \langle x, n\rangle = 0\}$



$\mathbb{R}^3$



Místo $\sphericalangle(\rho, \Gamma)$ počítáme

odchylku $\sphericalangle(\rho, [n])$

$$\sphericalangle(\rho, \Gamma) = \frac{\pi}{2} - \sphericalangle(\rho, [n])$$

Spočítáme kolmou projekci vektoru

$n = (1,2,-1,3,1)$ do $Z(\rho) = [(1,-1,1,1,3),$
$(1,-3,-3,-3,-9)]$
$= [v_1, v_2]$

$Pn = a v_1 + b v_2$

$n - Pn \perp v_1, v_2$

$$a\langle v_1, v_1\rangle + b\langle v_2, v_1\rangle = \langle u, v_1\rangle$$

$$a\langle v_1, v_2\rangle + b\langle v_2, v_2\rangle = \langle u, v_2\rangle$$

$$\left(\begin{array}{cc|c} 13 & -29 & 4 \\ -29 & 109 & -20 \end{array}\right) \sim \left(\begin{array}{cc|c} 13 & -29 & 4 \\ -3 & 51 & -12 \end{array}\right)$$

2.ř : 3

$$\sim \left(\begin{array}{cc|c} -1 & 17 & -4 \\ 0 & 192 & -48 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 17 & -4 \\ 0 & 4 & -1 \end{array}\right)$$

$$b = -\frac{1}{4} \qquad a = -\frac{1}{4}$$

$$P\vec{u} = -\frac{1}{4}(v_1 + v_2) = \frac{1}{2}(-1, 2, 1, 1, 3)$$

Odchylka $\measuredangle([\vec{u}], [P\vec{u}])$ je α

$$\cos\alpha = \frac{\|P\vec{u}\|}{\|u\|} = \frac{\frac{1}{2}\|(-1, 2, 1, 1, 3)\|}{\|(1, 2, -1, 3, 1)\|} = \frac{1}{2}$$

$$\alpha = \frac{\pi}{3}$$

Odchylka nadroviny Γ od roviny ρ

je $\frac{\pi}{2} - \alpha = \frac{\pi}{2} - \frac{\pi}{3} = \underline{\underline{\frac{\pi}{6}}}$

Další úlohy na procvičení

Příklad. 1. [Studijní materiály v ISu, domácí úkoly ke cvičení č. 7, úloha 2b.]
V $\mathbb{R}^4$ určete vzdálenost přímky p od roviny ρ

$$p : [1, 6, 2, 4] + r(2, -1, 2, -2),$$
$$\rho : x_1 + x_2 - x_3 + x_4 = 11, \quad x_1 + x_2 + 3x_3 + 3x_4 = 57.$$

a body $C \in p$ a $D \in \rho$, v nichž se tato vzdálenost realizuje.

Příklad. 2. [Studijní materiály v ISu, domácí úkoly ke cvičení č. 7, úloha 3]
V $\mathbb{R}^4$ určete vzdálenost rovin ρ a η a body, v nichž se realizuje.

$$\rho : [2, 0, -1, 3] + s(1, -2, 0, 1) + t(2, -3, -2, 3),$$
$$\eta : [2, -1, -2, 9] + p(3, 6, 6, -10) + q(4, 5, 4, -8).$$

Příklad. 3. [Studijní materiály v ISu, domácí úkoly ke cvičení č. 8, úloha 1a]
V $\mathbb{R}^4$ určete odchylku vektoru $u = (1, 1, 3, 5, 6)$ od podprostoru

$$V = [(1, 7, -1, -1, -6), (1, -5, 5, 5, 6)].$$