Konzultace 26. 3. 2021

kvadratická forma $g: U \to \mathbb{R}$

Je-li g kvadr. forma

$$g(u) = \underline{f(u,u)}$$ kde f je bilin. sym. forma

$$\underline{g(tu)} = f(tu,tu) = t\,f(u,tu) = t \cdot t\, f(u,u) = t^2 f(u,u) = \underline{t^2 g(u)} \qquad t \neq 1$$

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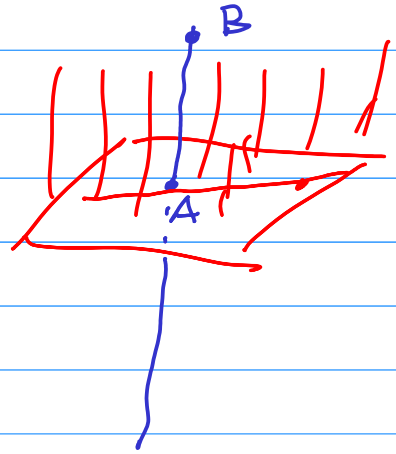
1. díl afinní geometrie

(1) $M = \{[x,y,z] \in \mathbb{R}^3;\ x + 2y + 3z \geq 4\} \subseteq \mathbb{R}^3$ ≥ poloprostor; = rovina

bod + vekt. podprostor

M je afinní podprostor $\iff \forall A,B \in M$, celá přímka AB leží v M.



rovina

$A = [4,0,0]$ na hranici

$B = [0,1,1]$ uvnitř

$$tA + (1-t)B = [4t,\ 1-t,\ 1-t]$$

$$4t + 2(1-t) + 3(1-t) = 5 - t$$

$t = 3$

$4 \mathcal{M}$

$3A + (-2)B = [12, -2, -2]$

$12 + 2(-2) + 3(-2) = 2 < 4$

$\mathcal{M}$ není af. podprostor.

(2) $\mathcal{M} = \{ X \in Mat_{2\times 2}(\mathbb{R}),\ (8\ 3) \cdot X = (20\ 21) \}$

$$(8\ 3) \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = (20\ 21)$$

$$\begin{aligned} &\overset{1}{8 x_{11}} + \overset{4}{3 x_{21}} = 20 \\ &\overset{0}{8 x_{12}} + \overset{7}{3 x_{22}} = 21 \end{aligned}$$

$$\mathcal{M} = \begin{pmatrix} 1 & 0 \\ 4 & 7 \end{pmatrix} + \{ Y \in Mat_{2\times 2}(\mathbb{R}),\ (8\ 3)\, Y = (0\ 0) \}$$

$\nearrow$ vekt. podprostor

$Z(\mathcal{M}) = \{ Y \in Mat_{2\times 2}(\mathbb{R})\ (8\ 3)\, Y = (0\ 0) \}$

$\dim \mathcal{M} = \dim Z(\mathcal{M})$

$= 4 - h(\text{matice}) \rightarrow \begin{pmatrix} 8 & 3 & 0 & 0 \\ 0 & 0 & 8 & 3 \end{pmatrix}$

$= 4 - 2 = \underline{\underline{2}}$

(3) $\mathcal{M} = \{ f \in \mathbb{R}^{\mathbb{R}};\ \exists n \in \mathbb{N}\ \forall x \geq n\ \ f(x) \geq x^2 \}$

$f(x) = x^2 \qquad f \in \mathcal{M} \qquad \forall x \geq 0 \quad f(x) \geq x^2$

$g(x) = 2x^2 \quad g \in M \quad \forall x \geq 0 \quad f(x) \leq 2x^2$

$t\, f(x) + (1-t)\, g(x) = t x^2 + (1-t)\, 2x^2 =$

$h(x) = 2x^2 - t x^2 \qquad t = 2$

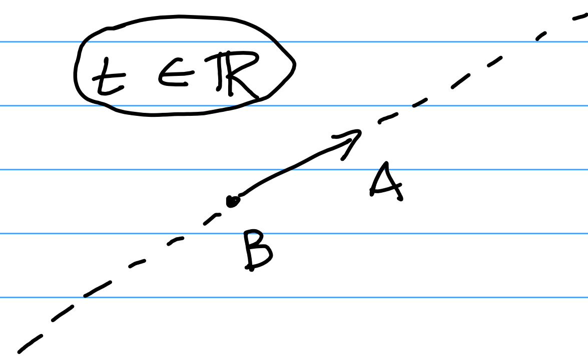
$h(x) = 2x^2 - 2x^2 = 0$

$\forall n \in \mathbb{N} \quad \exists x \geq n \quad \underset{= 0}{h(x)} < x^2$

$h \notin M$, M není afinní podmnožina.

$t A + (1-t) B \quad (t \in \mathbb{R})$

$= B + t(A-B)$



úsečka pro $t \in [0,1]$

2. příklad

Vzájemná poloha v $\mathbb{R}^4$

roviny $\rho = \{A + pu + qv,\ p, q \in \mathbb{R}\}$

$\sigma = \{A' + pu' + qv',\ p, q \in \mathbb{R}\}$

$A = [-6, -1, -2, 3] \quad u = (2, -3, 1, -2)$

$v = (3, -4, 6, -1)$

$A' = [1, -2, -3, 2], \quad u' = (2, -4, 0, -3)$

$v' = (1, 2, 3, 3)$

1. $\rho \cap \sigma = \{X, \dots\}$

$$X = A + pu + qv = A' + p'u' + q'v'$$

$$pu + qv - p'u' - q'v' = A' - A$$

$$\left(\begin{array}{cccc|c} 2 & 3 & -2 & -1 & 7 \\ -3 & -4 & 4 & -2 & -1 \\ 1 & 6 & 0 & -3 & -4 \\ -2 & -1 & 3 & -3 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 6 & 0 & -3 & -4 \\ 0 & -9 & -2 & 5 & 15 \\ 0 & 14 & 4 & -11 & -13 \\ 0 & 11 & 3 & -9 & -4 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 6 & 0 & -3 & -4 \\ 0 & 2 & 1 & -4 & 11 \\ 0 & 0 & -3 & 17 & -90 \\ 0 & -1 & 2 & -11 & 59 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 6 & 0 & -3 & -4 \\ 0 & -1 & 2 & -11 & 59 \\ 0 & 0 & -3 & 17 & -90 \\ 0 & 0 & 5 & -26 & * \\ 0 & 0 & 15 & -78 & * \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 6 & 0 & -3 & * \\ 0 & -1 & 2 & -11 & * \\ 0 & 0 & -3 & 17 & * \\ 0 & 0 & 0 & -3 & * \end{array}\right)$$

$-78 + 5 \cdot 17$

$\Rightarrow$ jediné řešení

$\rho \cap \sigma$ = jeden bod

$\Rightarrow \quad \cdot\ Z(\rho) \cap Z(\sigma) = \{\vec{o}\}$

ρ a σ jsou mimoběžné!

③ $\mathbb{R}^4$ přímka $q: [1,3,1,1] + t(-1,2,-1,1)$

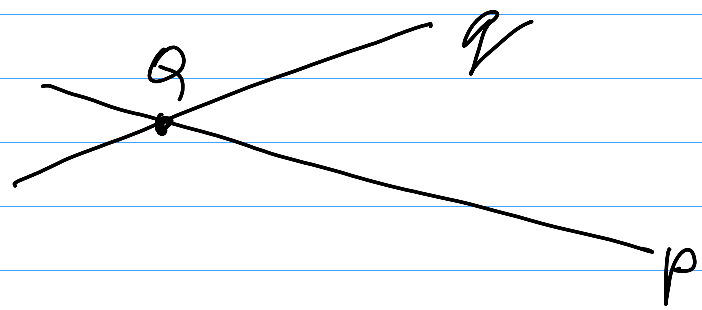
rovina ρ: $x_1 + x_2 - x_3 - 3x_4 = -2$

$2x_1 + x_2 - 2x_4 = 5$

Najít přímku p $\quad p \cap q \neq \emptyset$

$p \cap \rho \neq \emptyset$

p má směrový vektor $v = (1,1,2,0)$



$p \sqcup q = \alpha$ rovina

$\alpha : [1,3,1,1] + a(-1,2,-1,1) + b(1,1,2,0)$

$p \cap \rho \neq \emptyset \qquad p \cap \rho \subseteq \alpha \cap \rho \Rightarrow$

$\alpha \cap \rho \neq \emptyset$

Počítáme $\alpha \cap \rho$

α:
$$x_1 = 1 - a + b$$
$$x_2 = 3 + 2a + b$$
$$x_3 = 1 - a + 2b$$
$$x_4 = 1 + a$$

dosadíme do rovnic pro ρ

$$x_1 + x_2 - x_3 - 3x_4 = -2$$
$$1 - a + b + 3 + 2a + b - 1 + a - 2b - 3 - 3a = -2$$

$x_1 = 1$
$x_2 = 9$
$x_3 = 3$
$x_4 = 3$

$$-a = -2 \Rightarrow a = 2$$

$$2x_1 + x_2 - 2x_4 = 5$$
$$2-2a+2b + 3+2a+b - 2-2a = 5$$

$$-2a + 3b = 5-3 = 2$$
$$-4 + 3b = 2$$
$$3b = 6 \qquad \boxed{b = 2}$$

$$\rho \cap \alpha = \rho \cap p = R = [1, 9, 3, 3]$$

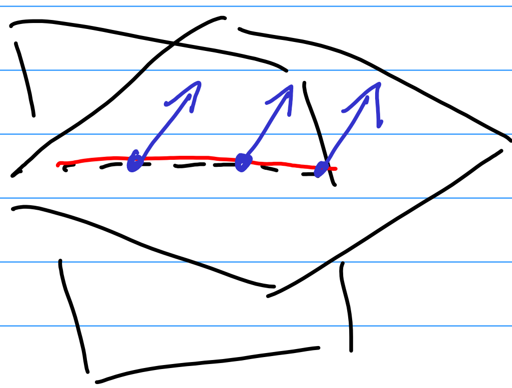
$$p : [1, 9, 3, 3] + c\,(1, 1, 2, 0)$$

$$t(\ ) - c(\ ) = [1,9,3,3] - [1,3,1,1]$$

$$p \cap q \quad [1, 9, 3, 3] + c\,(1, 1, 2, 0)$$
$$= [1, 3, 1, 1] + t(-1, 2, -1, 1)$$

$$\left(\begin{array}{cc|c} -1 & -1 & 0 \\ 2 & -1 & 6 \\ -1 & -2 & 2 \\ 1 & 0 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} t & c & \\ -1 & -1 & 0 \\ 0 & -3 & 6 \\ 0 & -3 & 2 \\ 0 & -1 & 2 \end{array}\right) \qquad \begin{array}{l} c = -2 \\ t = 2 \end{array}$$

$$p \cap q = Q = [-1, 7, -1, 3]$$



4. příklad

$\mathcal{M} = P + Z(\mathcal{M}) \qquad \mathcal{N} = Q + Z(\mathcal{N})$

Spočítat dim $\mathcal{M} \sqcup \mathcal{N}$ pomocí dimenzí $Z(\mathcal{M})$, $Z(\mathcal{N})$ a $Z(\mathcal{M}) \cap Z(\mathcal{N})$

Nejdříve se podívat na 2 přímky v $\mathbb{R}^3$

① p, q jsou mimoběžné

$$p : P + tu \qquad q : Q + sv$$

$$p \sqcup q = \mathbb{R}^3 \qquad p \sqcup q : P + [Q-P] + [u] + [v]$$

$$\dim (p \sqcup q) = \underset{=1}{\dim [u]} + \underset{=1}{\dim [v]} - \dim [u]_0 \cap [v] + \underset{=1}{\dim [Q-P]}$$

$$\boxed{\mathcal{M} \cap \mathcal{N} = \emptyset} \qquad Q - P \notin Z(\mathcal{M}) + Z(\mathcal{N})$$

$$Z(\mathcal{M} \sqcup \mathcal{N}] = [Q-P] + Z(\mathcal{M}) + Z(\mathcal{N})$$

$$\dim \mathcal{M} \sqcup \mathcal{N} = \underset{=1}{\dim (Q-P)} + \dim (Z(\mathcal{M}) + Z(\mathcal{N}))$$

$$= 1 + \dim Z(\mathcal{M}) + \dim Z(\mathcal{N}) - \dim (Z(\mathcal{M}) \cap Z(\mathcal{N}))$$

$p, q \subseteq \mathbb{R}^3$ rovnoběžné $\quad p \cap q = \emptyset$

$p \sqcup q = $ rovina

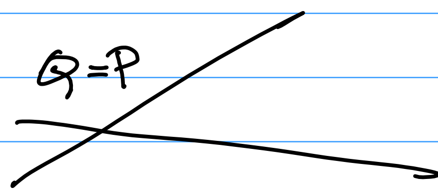
$$2 = \dim \text{rovina} = 1 + \underset{1}{\dim Z(p)} + \underset{1}{\dim Z(q)} - \dim \underbrace{Z(p) \cap Z(q)}_{1}$$

$p \cap q = \emptyset$

$Z(p) \cap Z(q) = Z(p)$

$[u] \cap [ku] = [u]$

p, q v $\mathbb{R}^3$ různoběžné



$p: \; P + [u]$

$q: \; P + [v]$

$p \sqcup q = \alpha$ rovina $\quad P + [u, v]$

$$2 = \dim \alpha = \dim Z(p) + \dim Z(q) - \dim \underset{0}{\big(Z(p) \cap Z(q)\big)} = 1 + 1 - 0$$

$m \cap n \ni A$

$m = A + Z(m) \quad n = A + Z(n)$

$m \sqcup n = A + Z(m) + Z(n)$

$$\dim \mathcal{M} \sqcup \mathcal{N} = \dim\left(Z(\mathcal{M}) + Z(\mathcal{N})\right)$$

$$= \dim Z(\mathcal{M}) + \dim Z(\mathcal{N})$$

$$- \dim\left(Z(\mathcal{M}) \wedge Z(\mathcal{N})\right)$$