LA- WEEK 11 JORDAN CANONICAL FORM

Unitary and selfadjoint operators on a space U have always a basis in U formed by eigenvectors. Generally, this is not true for other operators.

<u>Example</u> $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\varphi(x) = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

This operator has an eigenvalue 2 of algebraic multiplicity 2 and geometric multiplicity 1, since $\ker(\varphi - 2\,id) = [\begin{pmatrix} 1 \\ 0 \end{pmatrix}]$. So φ has not a basis in $\mathbb{R}^2$ formed by eigenvectors. That is why φ cannot have in any basis α

$$(\varphi)_{\alpha,\alpha} \quad \text{diagonal}.$$

<u>Motivation</u> For an operator φ with the property that the sum of algebraic multiplicities of its eigenvalues is equal to the dimension of the space on which φ is defined, we want to find a basis in which the matrix of φ is as simple as possible.

This simple form is <u>JORDAN CANONICAL FORM</u>.

(We will write JCF.)

Definition of JCF

(1) Jordan cell $J_k(\lambda)$ is the matrix $k \times k$

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & & & \\ 0 & \lambda & 1 & & 0 & \\ 0 & 0 & \lambda & & & \\ & & & \ddots & & \\ & 0 & & & \lambda & 1 \\ & & & & 0 & \lambda \end{pmatrix}$$

(2) A matrix is in Jordan canonical form if it is block diagonal with Jordan cells on the diagonal:

$$J = \begin{pmatrix} J_{k_1}(\lambda_1) & & & 0 \\ & J_{k_2}(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J_{k_p}(\lambda_p) \end{pmatrix}$$

Example

$$J = \begin{pmatrix} 2 & 1 & 0 & & & & & & & \\ 0 & 2 & 1 & & & & & 0 & & \\ 0 & 0 & 2 & & & & & & & \\ & & & 2 & 1 & & & & & \\ & & & 0 & 2 & & & & & \\ & & & & & 3 & & & & \\ & & & & & & 4 & 1 & 0 \\ & 0 & & & & & 0 & 4 & 1 \\ & & & & & & 0 & 0 & 4 \end{pmatrix}$$

Chain for the eigenvalue λ of ~~the~~ an operator $\varphi : U \to U$ is a sequence $u_1, u_2, \ldots u_k$ of non zero vectors from U such that

$$
\begin{aligned}
(\varphi - \lambda id)\, u_1 &= 0 \\
(\varphi - \lambda id)\, u_2 &= u_1 \\
(\varphi - \lambda id)\, u_3 &= u_2 \\
&\vdots \\
(\varphi - \lambda id)\, u_k &= u_{k-1}
\end{aligned}
$$

We will write it sometimes in the following way

$$u_k \xrightarrow{\varphi - \lambda id} u_{k-1} \longrightarrow \cdots \longrightarrow u_3 \xrightarrow{\varphi - \lambda id} u_2 \xrightarrow{\varphi - \lambda id} u_1 \xrightarrow{\varphi - \lambda id} 0 .$$

<u>Lemma</u> Let $u_1, u_2, \ldots u_k$ be a chain for the eigenvalue λ of an operator $\varphi : U \to U$. Then $V = [u_1, u_2, \ldots, u_k] \subseteq U$ is an invariant subspace with respect to φ, the vectors $u_1, u_2, \ldots u_k$ form a basis α of V and in this basis the matrix of $\varphi / V : V \to V$ is

$$(\varphi/V)_{\alpha,\alpha} = \begin{pmatrix} \lambda & 1 & 0 & & & \\ 0 & \lambda & 1 & & & \\ 0 & 0 & \lambda & & & \\ & & & \ddots & & \\ & & & & \lambda & 1 \\ & & & & & \lambda \end{pmatrix} = J_k(\lambda)$$

Proof (is useful for the computations)

The conditions from the definion of a chain can be rewritten in equivalent way:

$$(\varphi - \lambda\,\mathrm{id})\,u_1 = 0 \iff \varphi(u_1) = \lambda u_1 = \lambda u_1 + 0\cdot u_2 + 0\ldots$$

$$(\varphi - \lambda\,\mathrm{id})\,u_2 = u_1 \iff \varphi(u_2) = u_1 + \lambda u_2 + 0\,u_3 + \ldots$$

$$(\varphi - \lambda\,\mathrm{id})\,u_3 = u_2 \iff \varphi(u_3) = 0\cdot u_1 + u_2 + \lambda u_3 + 0\cdot u_4 \ldots$$

$$\ldots\ldots \qquad\qquad \ldots\ldots$$

Hence $\varphi(u_i) \in V$, and consequently $\varphi(V) \subseteq V$, so V is an invariant subspace.

The vectors $u_1, u_2, \ldots u_k$ are linearly independent (it can be shown by induction with respect to k, but I will not do it).

So $\alpha = (u_1, \ldots u_k)$ is a basis of V and in this basis

$$(\varphi/V)_{\alpha,\alpha} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$

Jordan Theorem

Let U be a vector space of dimension n and $\varphi : U \to U$ an operator. Suppose that the sum of algebraic multiplicities of all eigenvalues of φ is equal to n. Then there is a basis α in U such that

$$(\varphi)_{\alpha,\alpha} = J$$

is a matrix in Jordan canonical form. This matrix is determined uniquely up to the order of Jordan cells.

<u>Remark</u> The basis α from theorem above is formed by chains to eigenvalues of the operator φ.

<u>Jordan Theorem over complex numbers</u>

If $\varphi : U \to U$ and U is a complex vector space, we can omit the assumption on the multiplicities, since it is always satisfied. (Every polynomial over $\mathbb{C}$ of degree n has n roots including multiplicities.)

Matrix version of Jordan Theorem

Let A be an $(n\times n)$-matrix over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Suppose that the sum of multiplicities of its eigenvalues is equal to n. Then A is similar to a matrix J in JCF, i.e. there is a regular matrix P such that

$$J = P^{-1}AP.$$

The matrix J is determined uniquely up to the order of Jordan cells.

Remark 1 The matrix P is not determined uniquely!

2 Over $\mathbb{C}$ we can omit the assumption.

Proof : Jordan Theorem $\Rightarrow$ Matrix version

Let A be an $(n\times n)$-matrix with elements in $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Then it defines an operator

$$\varphi : \mathbb{K}^n \to \mathbb{K}^n \quad \varphi(x) = Ax$$

which satisfies the assumption of Jordan Theorem. Then there is a basis α in $\mathbb{K}^n$ such that $(\varphi)_{\alpha,\alpha} = J$ is a matrix in JCF.

It holds

$$J = (\varphi)_{\alpha,\alpha} = (id)_{\alpha,\varepsilon}(\varphi)_{\varepsilon,\varepsilon}\,(id)_{\varepsilon,\alpha} = P^{-1}AP$$

where $\varepsilon = (e_1, e_2, \ldots, e_n)$ is the standard basis.

FINDING JCF FOR OPERATORS AND MATRICES IN DIMENSION 3 AND 4

$\boxed{\text{Rule 1}}$ On the diagonal of JCF there are eigenvalues of our operator φ (matrix A), every as many times as its ~~...~~ algebraic multiplicity.

Example 1a $\varphi : \mathbb{R}^3 \to \mathbb{R}^3 \quad \varphi(x) = Ax$

$$A = \begin{pmatrix} 3 & 5 & 3 \\ -4 & -9 & -6 \\ 6 & 15 & 10 \end{pmatrix}$$

Characteristic polynomial is

$$\det(A - \lambda E) = (2-\lambda)(1-\lambda)^2$$

Eigenvalues

$\lambda_1 = 2$, alg. multiplicity 1, geom. multiplicity 1
eigenvector $v_1 = (1, -2, 3)^T$

$\lambda_2 = 1$, alg. mult. 2, geom. mult. 2
eigenvectors $v_2 = (3, 6, -8)^T$
$v_3 = (1, -1, 1)^T$

In the basis $\alpha = (v_1, v_2, v_3)$

$$(\varphi)_{\alpha,\alpha} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is a matrix in JCF with 3 cells of dimension 1.

<u>Example 1b</u> $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ $\varphi(x) = Bx$

$$B = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Char. polynomial is $(2-\lambda)(1-\lambda)^2$

Eigenvalues

$\lambda_1 = 2$ alg. mult. 1, geom. mult. 1
eigenvector $v_1 = (1,0,0)^T$

$\lambda_2 = 1$ alg. mult. 2, geom. mult. 1
eigenvector $v_2 = (1,-1,0)$

According to Rule 1 the diagonal of JCF is $\begin{matrix} 2 & & \\ & 1 & \\ & & 1 \end{matrix}$, but JCF cannot have three cells, otherwise the geom. mult. of $\lambda_2 = 1$ would be 2.

So JCF has to be

$$J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

We are looking for a basis α in which

$$(\varphi)_{\alpha,\alpha} = J = \left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

For the eigenvalue 2 we have eigenvector v_1 in α. For the eigenvalue 1 we have to find a chain of the length 2 for eigenvalue 1.

The first vector of this chain is an eigenvector v_2. The second vector is v_3 such that

$$(B - 1\cdot E)\, v_3 = v_2$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

In the basis $\alpha = (v_1, v_2, v_3)$

$$(\varphi)_{\alpha,\alpha} = \left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

Moreover

$$(\varphi)_{\alpha,\alpha} = (id)_{\alpha,\varepsilon}\, (\varphi)_{\varepsilon,\varepsilon}\, (id)_{\varepsilon,\alpha}$$

$$J = P^{-1} \quad B \quad P$$

where $P = (id)_{\varepsilon,\alpha} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1/2 \\ 0 & 0 & -1/2 \end{pmatrix}.$

Rule 2

The number of Jordan cells for the eigenvalue λ is equal to the geometric multiplicity of λ.

Explanation: Chain

$$v_k \xrightarrow{\varphi-\lambda id} v_{k-1} \cdots \longrightarrow v_2 \xrightarrow{\varphi-\lambda id} \underbrace{v_1 \xrightarrow{\varphi-\lambda id} 0}_{v_1 \text{ is an eigenvalue}}$$

Example 2 $\quad \varphi : \mathbb{R}^3 \to \mathbb{R}^3 \quad \varphi(x) = Ax$

$$A = \begin{pmatrix} 13 & -28 & 3 \\ 4 & -8 & 1 \\ -1 & 4 & 1 \end{pmatrix}$$

Char. polynomial is $(2-\lambda)^3$

Eigenvalue $\lambda = 2$ of alg. mult. 3 and geom. mult. 1

Eigenvector $u_1 = (2,1,2)^T$.

According to Rule 2, JCF has only one cell, so JCF has to be

$$J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

That is why we have to look for a basis α in the form of a chain of length 3 starting with the vector u_1.

We solve equations

$$(A-2E)\,u_2 = u_1$$

$$(A-2E)\,u_3 = u_2$$

One of possible solutions is

$$u_2 = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

In the basis $\alpha = (u_1, u_2, u_3)$

$$(\varphi)_{\alpha,\alpha} = J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

<u>Example 3</u> $\varphi : \mathbb{R}^3 \to \mathbb{R}^3 \quad \varphi(x) = Ax$

$$A = \begin{pmatrix} 4 & 4 & 0 \\ -1 & 0 & 0 \\ -2 & -4 & 2 \end{pmatrix}$$ Char. polynomial is $(2-\lambda)^3$.

Eigenvalue 2 of alg. mult. 3 and geom. mult. 2

$\ker\,(A-2E) = [u, v]$ where

$$u = (2,-1,0)^T, \quad v = (0,0,1)^T$$

According to Rules 1 and 2 a JCF for φ contains two cells, one of dimension 1, the other of dimension 2

$$J = \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 2 \end{array}\right)$$

We want to find a basis α which consists of two chains for the eigenvalue 2, one of length 2 and one of length {1} (only eigenvector), which are linearly independent.

We have to find an eigenvector which is at the beginning of the chain of length 2.

We look for it in the form

$$au + bv$$

as a vector for which the equation

$$(A-2E)x = au + bv$$

has a solution.

Corresponding matrix is

$$\left(\begin{array}{ccc|c} 2 & 4 & 0 & 2a \\ -1 & -2 & 0 & -a \\ -2 & -4 & 0 & b \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 4 & 0 & 2a \\ -1 & -2 & 0 & -a \\ 0 & 0 & 0 & 2a+b \end{array}\right)$$

A solution exists if and only if $2a+b=0$.

Choose $a=1$, $b=-2$, $w_1 = au+bv = u-2v = (2,-1,-2)^T$ and compute w_2 as a solution of

$$(A-2E)w_2 = w_1.$$

One possibility is $w_2 = (-1, 1, 1)^T$.

We take $\alpha = (\underbrace{w_1, w_2}_{\text{a chain of length 2}}, v)$ an eigenvector which is not a multiple of w_2, i.e $v = (0,0,1)^T$)

In this basis $(\varphi)_{\alpha,\alpha} = \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 2 \end{array}\right)$.

<u>Homework</u> Find a basis α in which the operator $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$, $\varphi(x) = Dx$ has $(\varphi)_{\alpha,\alpha}$ in JCF.

$$D = \begin{pmatrix} 6 & -9 & 5 & 4 \\ 7 & -13 & 8 & 7 \\ 8 & -17 & 11 & 8 \\ 1 & -2 & 1 & 3 \end{pmatrix}$$

<u>Hint</u>: Eigenvalues are 2 of alg.mult. 3 and 1.