LA - WEEK 12 JORDAN CANONICAL FORM II

Last time we derived two rules for computation of Jordan canonical forms (JCT).

<u>Rule 1</u> On the diagonal of JCF there are eigenvalues of our operator φ (matrix A), every as many ~~as~~ times as its algebraic multiplicity.

<u>Rule 2</u> The number of Jordan cells with the eigenvalue λ is equal to the geometric multiplicity of λ.

These two rules are sufficient for computations in dimensions 2 and 3, but not in higher ones.

If we have an eigenvalue λ of alg. multiplicity 4 and geometric multiplicity 2, the corresponding Jordan cells can be

either

$$J_1 = \left(\begin{array}{cc|cc} \lambda & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & \lambda & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \lambda & 1 \\ & & 0 & \lambda \end{array}\right)$$

There is no chain of length 3 here.

or

$$J_2 = \left(\begin{array}{ccc|c} \lambda & 1 & 0 & \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & \\ \hline & 0 & & \lambda \end{array}\right)$$

There is a chain of legth 3 here

②

$$J_1 - \lambda E = \left(\begin{array}{cc|cc} 0 & 1 & & \\ 0 & 0 & & \\ \hline & & 0 & 1 \\ & & 0 & 0 \end{array}\right) \qquad J_2 - \lambda E = \left(\begin{array}{ccc|c} 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \\ \hline & & & 0 \end{array}\right)$$

$$(J_1 - \lambda E)^2 = 0 \qquad (J_2 - \lambda E)^2 = \left(\begin{array}{ccc|c} 0 & 0 & 1 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ \hline & & & 0 \end{array}\right)$$

If we have a matrix A of the form 4×4 with an eigenvalue λ of alg. multiplicity 4 and geom. multiplicity 2, how to decide if is similar to J_1 or J_2?

① $A = Q^{-1} J_1 Q$

$$(A - \lambda E) = Q^{-1} J_1 Q - \lambda E = Q^{-1}(J_1 - \lambda E) Q$$

$$(A - \lambda E)^2 = \left[Q^{-1}(J_1 - \lambda E) Q\right]\left[Q^{-1}(J_1 - \lambda E) Q\right] =$$

$$= Q^{-1}(J_1 - \lambda E)^2 Q = Q^{-1} \cdot 0 \cdot Q = 0$$

② $A = Q^{-1} J_2 Q$

$$(A - \lambda E)^2 = Q^{-1} \underbrace{(J_2 - \lambda E)^2}_{\neq 0} Q \neq 0.$$

Example 4 $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ $\varphi(x) = Ax$

$$A = \begin{pmatrix} -13 & 5 & 4 & 2 \\ 0 & -1 & 0 & 0 \\ -30 & 12 & 9 & 5 \\ -12 & 6 & 4 & 1 \end{pmatrix}$$

Char. polynomial $(1+\lambda)^4$
Eigenvalue $\lambda = -1$
of alg. multiplicity 4
and geom. mult. 2

Eigenvectors $u = (1,0,3,0)^T$, $v = (0,0,1,-2)^T$

We want to find a chain of length 2 or 3.
First we are looking for $a, b \in \mathbb{R}$ such that the equation

$$(A+E)\,w = au + bv$$

has a solution. We will find that the equation has a solution for every $a, b \in \mathbb{R}$.
Hence a chain of length 2 can start with the eigenvector u and also with the linear independent vector v.
So there are two linearly independent chains of length two. That is why JCF for the operator φ has two cells 2×2.

$$J = \left(\begin{array}{cc|cc} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

We can compute the chains

$$(A+E)\bar{u} = u \qquad\qquad (A+E)\bar{v} = v$$

$$\bar{u} = (0,-1,0,3)^T + au + bv \qquad \bar{v} = (0,-2,0,5)^T + cu + dv$$

Taking $a=b=c=d=0$ we get a basis

$$\alpha = (u, \bar{u}, v, \bar{v})$$

which consists of both chains. In this basis

$$(\varphi)_{\alpha,\alpha} = J = \left(\begin{array}{cc|cc} -1 & 1 & & \\ 0 & -1 & & \\ \hline & & -1 & 1 \\ & & 0 & -1 \end{array}\right)$$

For the standard basis $\varepsilon = (e_1, e_2, e_3, e_4)$ we get

$$(\varphi)_{\alpha,\alpha} = (id)_{\alpha,\varepsilon}\ (\varphi)_{\varepsilon,\varepsilon}\ (id)_{\varepsilon,\alpha}$$

$$J = P^{-1} \cdot A \cdot P$$

where $P = (id)_{\varepsilon,\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -2 \\ 3 & 0 & 1 & 0 \\ 0 & 3 & -2 & 5 \end{pmatrix}$.

Choosing different coefficients a, b, c, d we get diffent bases and different matrices P with the same property.

Example 5 $\varphi: \mathbb{R}^4 \to \mathbb{R}^4$, $\varphi(x) = Ax$,

$$A = \begin{pmatrix} 4 & 3 & 2 & -3 \\ 6 & 9 & 4 & -8 \\ -3 & -4 & -1 & 4 \\ 9 & 9 & 6 & -8 \end{pmatrix}$$

Char. polynomial is $(1-\lambda)^4$. Eigenvalue $\lambda = 1$ of alg. mult. 4 and geom. mult. 2

Eigenvectors $u = (0,1,0,1)^T$, $v = (-2,0,3,0)^T$

For which $a, b \in \mathbb{R}$ has the system of linear equations

$$(A - E)\, w = au + bv$$

a solution?

$$\left(\begin{array}{cccc|c} 3 & 3 & 2 & -3 & -2b \\ 6 & 8 & 4 & -8 & a \\ -3 & -4 & -2 & 4 & 3b \\ 9 & 9 & 6 & -9 & a \end{array}\right) \sim \left(\begin{array}{cccc|c} 3 & 3 & 2 & -3 & -2b \\ 0 & 2 & -2 & 10 & a+4b \\ 0 & -1 & 0 & 1 & b \\ 0 & 0 & 0 & 0 & a+6b \end{array}\right)$$

The system has a solution if and only if $\underline{a + 6b = 0}$.

Choose $a = -6$, $b = 1$. $-6u + v = (-2, -6, 3, -6)^T$.

A solution of the system $(A-E)w = au + bv$ is

$$w = \left(\tfrac{1}{3}, -1, 0, 0\right)^T + a_1 u + b_1 v$$

This will be the second vector of a chain of ~~[illegible]~~ the length 3. We are looking for $a_1, b_1 \in \mathbb{R}$ such that the system

$$(A-E)\,z = \left(\tfrac{1}{3}, -1, 0, 0\right)^T + a_1 u + b_1 v$$

has a solution.

$$\left(\begin{array}{cccc|c} 3 & 3 & 2 & -2 & \frac{1}{3} - 2b_1 \\ 6 & 8 & 4 & -8 & -1 + a_1 \\ -3 & -4 & -2 & 4 & 3b_1 \\ 9 & 9 & 6 & -9 & a_1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 3 & 3 & 2 & -3 & \frac{1}{3} - 2b_1 \\ 0 & 2 & -2 & 10 & -1 + a_1 \\ 0 & -1 & 0 & 1 & 3b_1 \\ 0 & 0 & 0 & 0 & -1 + a_1 + 6b_1 \end{array}\right)$$

The system has a solution if and only if

$$-1 + a_1 + 6b_1 = 0.$$

Choose $a_1 = 1$, $b_1 = 0$. Then

$$w = \left(\tfrac{1}{3}, 0, 0, 1\right)^T \qquad z = \left(0, 0, \tfrac{2}{3}, \tfrac{1}{3}\right)^T$$

$$(+\,cu + dv)$$

A chain of the length 3 is

$$(-6u + v,\ w,\ z)$$

The basis $\alpha = (\underbrace{6u + v,\ w,\ z}_{\text{a chain of length 3}},\ \overbrace{u}^{\text{a chain of length 1}})$

has the property that

$$(\varphi)_{\alpha,\alpha} = \left(\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \\ \hline & 0 & & 1 \end{array}\right)$$

A different solution which is possible only in the cases when all theeigenvalues are the same.

Choose a vector u_3 and compute $u_2 = (A-E)u_3$. If $u_2 \neq \vec{0}$ compute $u_1 = (A-E)u_2$. If $u_1 \neq 0$, then u_1, u_2, u_3 form a chain of length 3.

$$u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \xmapsto{A-E} u_2 = \begin{pmatrix} 3 \\ 6 \\ -3 \\ 9 \end{pmatrix} \xmapsto{A-E} u_1 = \begin{pmatrix} -6 \\ -18 \\ 9 \\ -18 \end{pmatrix} \xmapsto{A-E} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking $\mathcal{B} = \left(u_1, u_2, u_3, u_4 = u = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right)$

we have

$$(\varphi)_{\mathcal{B},\mathcal{B}} = \left(\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & \\ \hline & 0 & & 1 \end{array}\right).$$

Homework 12A Find a Jordan canonical form of the matrix

$$F = \begin{pmatrix} 3 & -1 & 1 & -7 \\ 9 & -3 & -7 & -1 \\ 0 & 0 & 4 & -8 \\ 0 & 0 & 2 & -4 \end{pmatrix}$$

and a matrix P such that

$$J = P^{-1} F P.$$

Homework 12 B Find a JCF J of the matrix

$$N = \begin{pmatrix} 4 & 3 & 2 & -3 \\ 6 & 9 & 4 & -8 \\ -3 & -4 & -1 & 4 \\ 9 & 9 & 6 & -8 \end{pmatrix}$$

and a matrix P such that

$$J = P^{-1} N P.$$

(Hint: eigenvalue 1 of alg. multiplicity 4)

Homework 12C Find a JCF J of the matrix

$$D = \begin{pmatrix} 6 & -9 & 5 & 4 \\ 7 & -13 & 8 & 4 \\ 8 & -17 & 11 & 8 \\ 1 & -2 & 1 & 3 \end{pmatrix}$$

and a matrix P such that

$$J = P^{-1} D P.$$

(Hint: eigenvalue 2 of alg. multiplicity 3 and an eigenvalue 1)

Note: Homework 12C = Homework 11