Exercise sessions for the Set theory course, spring semester 2021

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Week 1

Definition Given some symbols $p, q, \dots$ and a (non-quantified) formula containing the given symbols and $\rightarrow, \neg, =$, giving a truth table means to specify the truth values of the whole formula for each possible assignment of truth values on the symbols $p, q, \dots$.
We say that two formulas are equivalent if they have the same truth tables.

Example The formula $p \rightarrow q$ has the following truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Recall that we can shorten $\neg p \rightarrow q$ into $p \vee q$ and $\neg(p \rightarrow \neg q)$ into $p \wedge q$. Let us compute their truth tables and see that they correspond with the expected ones, that is, $p \vee q$ is false if and only if both p and q are false, and $p \wedge q$ is true if and only if both p and q are true.

p	q	$\neg p$	$\neg p \rightarrow q$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

p	q	$\neg q$	$p \rightarrow \neg q$	$\neg(p \rightarrow \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	T	F
F	F	T	T	F

Exercise Find the truth table of the formula

$$\neg(p \vee q) \vee (\neg p \wedge q).$$

Solution We can build it one step at a time in the following table

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg p \wedge q$	$\neg(p \vee q) \vee (\neg p \wedge q)$
T	T	T	F	F	F	F
T	F	T	F	F	F	F
F	T	T	F	T	T	T
F	F	F	T	T	F	T

Exercise Verify if the two following formulas are equivalent:

$$\neg(p \wedge q) \wedge (\neg q \wedge \neg r)$$
$$\neg p \wedge \neg q \wedge \neg r.$$

Solution We need to calculate the truth tables of both formulas:

p	q	r	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$	$\neg r$	$\neg q \wedge \neg r$	$\neg(p \wedge \neg q) \wedge (\neg q \wedge \neg r)$
T	T	T	F	F	T	F	F	F
T	T	F	F	F	T	T	F	F
T	F	T	T	T	F	F	F	F
T	F	F	T	T	F	T	T	F
F	T	T	F	F	T	F	F	F
F	T	F	F	F	T	T	F	F
F	F	T	T	F	T	F	F	F
F	F	F	T	F	T	T	T	T

p	q	r	$\neg p$	$\neg q$	$\neg r$	$\neg p \wedge \neg q \wedge \neg r$
T	T	T	F	F	F	F
T	T	F	F	F	T	F
T	F	T	F	T	F	F
T	F	F	F	T	T	F
F	T	T	T	F	F	F
F	T	F	T	F	T	F
F	F	T	T	T	F	F
F	F	F	T	T	T	T

so they are equivalent, because their truth values always coincide.

Exercise Suppose we are given a language with a binary relation symbol R. Express in formulas the following properties:

1. R is a function;
2. R is a bijection;
3. R is an equivalence relation;
4. R is a linear order on a set with three elements.

Solution We will have to be specifically careful to all instances of the phrase "it exists a unique x such that $\phi(x)$". The usual way to express this is

$$\exists! x \phi(x)$$

but using the symbols of predicate logic this has to be written

$$\exists x \phi(x) \wedge \forall y (\phi(y) \rightarrow y = x).$$

Now let us proceed with the solution

1. $\forall x(\exists y Rxy \wedge \forall z(Rxz \rightarrow z = y))$;
2. $\forall x(\exists y Rxy \wedge \forall z(Rxz \rightarrow z = y)) \wedge \forall y(\exists x Rxy \wedge \forall w(Rwy \rightarrow w = x))$;
3. $\forall x \forall y \forall z(Rxx) \wedge (Rxy \rightarrow Ryx) \wedge ((Rxy \wedge Ryz) \rightarrow Rxz)$;
4. $\exists x \exists y \exists z(\forall w(w = x) \vee (w = y) \vee (w = z))\wedge$
   $(\neg x = y \wedge \neg y = z \wedge \neg x = z)\wedge$
   $(Rxx \wedge Rxy \wedge Ryy \wedge Ryz \wedge Rzz \wedge Rxz)$
   $\forall u \forall v(Ruv \rightarrow (u = v \vee (u = x \wedge v = y) \vee (u = x \wedge v = z) \vee (u = y \wedge v = z)))$.

Exercise Let us consider a language with one binary operation symbol $+$. Express the following properties of a realization M in formulas:

1. M is associative;
2. M contains a unit element;
3. M is a group;
4. M is an abelian group.

Solution As is usual practice, the symbol $+$ goes between variables instead of preceding them. If we want to stick to our definition of terms and formulas, let us for example use the symbol f for the sum operation, and write it in prefix notation.

1. $\forall a \forall b \forall c ffabc = fafbc$;
2. $\exists e \forall a(fae = a) \wedge (fea = a)$;
3. This includes the two previous formulas plus invertibility of all elements:
   $$\forall a \exists b(fab = e) \wedge (fba = e);$$
4. This includes all of the previous plus
   $$\forall a \forall b fab = fba.$$
   Note that, if we are describing an abelian group, the existence of a unit and invertibility of elements can be simplified to the following forms:
   $$\exists e \forall a fae = a;$$
   $$\forall a \exists b fab = e.$$

Week 2

Definition Fixing a language, a sequent is a string of the form $\Theta \Rightarrow \phi$, where Θ is a finite set of formulas in the given language, ϕ is a single formula and $\Rightarrow$ is a new symbol, not comprised in the language.

Remark The way to think of sequents is: if all the premises in the antecedent Θ are true, then we conclude that ϕ is also true.
We can also take Θ to be an empty set, in which case we often abbreviate the sequent to the single formula ϕ.

Definition A theory is a pair $(\mathcal{L}, A)$, where $\mathcal{L}$ is a formal language and A is a set of sequents, called the axioms of set theory.

Informal definition A model for a theory is a realization that makes all its axioms true.

Example Given the theory whose language contains one binary operation symbol f, and having the two axioms

$$\forall a \forall b \forall c f f abc = fafbc;$$
$$\exists e \forall a (fae = a) \wedge (fea = a)$$

a model for this theory is precisely a monoid.

Example Given the theory whose language contains one binary operation symbol f and one constant symbol e, and having the three axioms

$$\forall a \forall b \forall c f f abc = fafbc;$$
$$\forall a (fae = a) \wedge (fea = a);$$
$$\forall a \exists b (fab = e) \wedge (fba = e)$$

then a model for this theory precisely a group.

Example Axiomatic set theory can be formalized as a theory in the sense above. Unfortunately, we will leave its description incomplete for now, concentrating on its parts a little at a time.
The language of set theory only comprises one binary relation symbol $\in$.
So far, we have seen only some of its axioms, namely:

Extensionality $\forall x \forall y (x = y \leftrightarrow (\forall z (z \in x \leftrightarrow z \in y)));$

Union $\forall x \exists y \forall z (z \in y \leftrightarrow (\exists t (t \in x \wedge z \in t)));$

Power $\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x);$

Pairing $\forall x \forall y \exists z (x \in z \wedge y \in z);$

Infinity $\exists x (\emptyset \in x \wedge \forall y (y \in x \to y \cup \{y\} \in x));$

note that when we write $\emptyset$, we can either define it through separation (see below) or add it to the language as a constant symbol, but in that case we need an additional axiom giving it the desired property.
The separation schema, rather than an axiom, is a family of axioms (in fact, an infinite family) indexed by all possible formulas in the given language. In order to define it precisely, we need a preliminary definition.

Definition Let x be a variable symbol and ϕ a formula in which x appears. Then x is said to be free in ϕ if its first occurrence is not quantified, i.e. it is not preceded by $\forall$ or $\exists$.
Given a formula $\phi(x_1, \ldots, x_n)$ where the variables x_i's are free, and given other variable symbols $a_1, \ldots, a_n$, then $\phi(a_1, \ldots, a_n)$ will denote the formula ϕ where all the occurrences of the variable symbols x_i's have been replaced by the corresponding symbols a_i's.

Continuation We can now define the axioms of separation. Given any formula $\phi(x)$ in the language of set theory, where x is free, then there is an axiom

$$\forall z \exists y (a \in y \leftrightarrow (a \in z \wedge \phi(a)).$$

Exercise Given a set A, how do we define the intersection of all its elements using the axioms of set theory?

Solution We choose one of the sets $a \in A$, and then we observe that the expression $\forall b \in A(x \in b)$ is a formula in which x is free. The intersection is now defined using separation, as

$$\cap_{b \in A} b := \{x \in a | \forall b \in A(x \in b)\}.$$

Remark We might want to define union similarly. The string $\exists b \in A(x \in b)$ is a formula in which x is free, and it certainly expresses the desired property of the union of all sets $b \in A$. However, there is a priori no set containing all the b's, which is precisely why we need the axiom of union.

- We know Cantor-Bernstein theorem, which states that if A and B are two sets and there are two injective functions $f : A \to B$ and $g : B \to A$, then there is a bijection $A \cong B$. The proof is non-trivial, but it is presented in the course notes, except for Tarski's fixed point thorem, which is used to find a subset $C \subseteq A$ such that $C = A - g(B - f(C))$.

Definition A complete lattice is a partially ordered set such that every set of points in it admits a join.

Tarski's theorem Let X be a complete lattice and $f : X \to X$ an order-preserving function. Then there is a fixed point for f, i.e. $\exists x f(x) = x$.

Proof By the axiom of separation, we can define $Y = \{y \in X | y \le f(y)\}$. Now take $x := \sup Y$, which exists because X is complete. We have then that $\forall y \in Y, y \le f(y) \le f(x)$ because f is order-preserving, therefore by definition of supremum we obtain $x \le f(x)$.
Moreover, this relation implies, again since f is order-preserving, that $f(x) \le f^2(x)$, which means that $f(x) \in Y$. Consequently, $f(x) \le x$. Now we have that $x \le f(x)$ and $f(x) \le x$ which immediately gives $x = f(x)$.

Remark Tarski's theorem can be applied in the proof of Cantor-Bernstein theorem, because for every set A, the power set $P(A)$ is a complete lattice, ordered by inclusion.

Week 3

Definition Recall that the product of two cardinals $\alpha = |A|$ and $\beta = |B|$ is defined as $\alpha \cdot \beta = |A \times B|$, and their exponential is $\alpha^\beta = |A^B|$, where the latter is the set of all functions $B \to A$.

Exercise Put the following cardinals in increasing order: $\aleph_0, 2 \cdot \aleph_0, \aleph_0 \cdot 2, 2^{\aleph_0}, \aleph_0^{\aleph_0}, c^{\aleph_0}, \aleph_0^c, \aleph_0 \cdot \aleph_0$.

Solution We know that $\aleph_0 \le 2 \cdot \aleph_0 \le \aleph_0 \cdot \aleph_0 = \aleph_0$, and moreover multiplication of cardinals is commutative, because for any two sets A and B the products $A \times B$ and $B \times A$ are naturally in bijection. Therefore we have

$$\aleph_0 = 2 \cdot \aleph_0 = \aleph_0 \cdot 2 = \aleph_0 \cdot \aleph_0.$$

We also know, by Cantor theorem, that $2^{\aleph_0} > \aleph_0$, and certainly $\aleph_0^{\aleph_0} \geq 2^{\aleph_0}$ so

$$\aleph_0^{\aleph_0} > \aleph_0.$$

Next, we can show that $\aleph_0^{\aleph_0} = 2^{\aleph_0} = c$. To see this, compute

$$2^{\aleph_0} \leq \aleph_0^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}.$$

Similarly we compute

$$2^{\aleph_0} \leq c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$$

so that $c^{\aleph_0} = c$.
Finally, we compute

$$\aleph_0^c \leq 2^{\aleph_0 \cdot c} = 2^c \leq \aleph_0^c$$

so $\aleph_0^c = 2^c > c$. In conclusion, we can order

$$\aleph_0 = 2 \cdot \aleph_0 = \aleph_0 \cdot 2 = \aleph_0 \cdot \aleph_0 < 2^{\aleph_0} = \aleph_0^{\aleph_0} = c^{\aleph_0} < \aleph_0^c.$$

Exercise Show that, for two infinite cardinals α, β, we have $\alpha + \beta = \alpha \cdot \beta = \max\{\alpha, \beta\}$.

Solution Let us first consider the sum, assuming without loss of generality that $\alpha \leq \beta$, so that $\max\{\alpha, \beta\} = \beta$. We compute

$$\beta \leq \alpha + \beta \leq \beta + \beta = 2 \cdot \beta = \beta.$$

The computation for the product is entirely analogous:

$$\beta \leq \alpha \cdot \beta \leq \beta \cdot \beta = \beta^2 = \beta.$$

Exercise Show that, for two infinite cardinals $\alpha \leq \beta$, we have $\alpha^\beta = 2^\beta$.

Solution We have to show that $2^\beta \leq \alpha^\beta$ and $\alpha^\beta \leq 2^\beta$. We can proceed as in the previous exercise.

$$2^\beta \leq \alpha^\beta \leq (2^\alpha)^\beta = 2^{\alpha \cdot \beta} = 2^\beta$$

Exercise Given infinite cardinals α, β, γ such that $\alpha < \beta$, determine if it is true that $\alpha^\gamma < \beta^\gamma$ and that $\gamma^\alpha < \gamma^\beta$.

Solution Neither of the two inequalities is true in general. For example, it is true that $\aleph < c$ but

$$\aleph_0^{\aleph_0} = c^{\aleph_0} = c$$

as we have seen above, and

$$(2^c)^{\aleph_0} = (2^c)^c$$

because they both equal (2^c), as it is easily proven with an argument just like the ones seen above.
In general, we can only say that $\alpha^\gamma \leq \beta^\gamma$ and $\gamma^\alpha \leq \gamma^\beta$.

Exercise Show that $|\mathbb{Q}| = |\mathbb{N}|$.

Solution We know that $|\mathbb{Z}| = |\mathbb{N}|$, so it suffices to show that $\mathbb{Q}^+$ is in bijection with $\mathbb{N}$. By the tabular argument we build a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$, that is, we enumerate $\mathbb{N}$ as in the following diagram

$$\begin{tikzcd}
(0,0) \arrow[r] & (0,1) \arrow[dl] & (0,2) \arrow[r] & \dots \arrow[dl] \\
(1,0) \arrow[d] & (1,1) \arrow[ur] & (1,2) \arrow[dl] & \dots \\
(2,0) \arrow[ur] & (2,1) \arrow[dl] & (2,2) & \dots \\
\vdots & \vdots & \vdots & \ddots
\end{tikzcd}$$

Now, we have a surjection $\mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$ given by $(p,q) \mapsto \frac{p}{q}$. Choosing p and q to be coprime gives us a left inverse function, which is clearly an injection $\mathbb{Q}^+ \hookrightarrow \mathbb{N} \times \mathbb{N} \cong \mathbb{N}$.
Moreover, there is an injection $\mathbb{N} \hookrightarrow \mathbb{Q}^+$ given simply by $n \mapsto \frac{n}{1}$.
We conclude using Cantor-Bernstein's theorem, which provides a bijection $\mathbb{N} \cong \mathbb{Q}^+$.

Exercise Show that $|\mathbb{R}| > |\mathbb{N}|$.

Solution There is an inclusion $\mathbb{N} \hookrightarrow \mathbb{R}$, so $|\mathbb{N}| \leq |\mathbb{R}|$. We need to prove that there is no bijection $\mathbb{R} \to \mathbb{N}$.
By a geometrical projection, we know that the open interval $(0,1)$ is in bijection with the whole $\mathbb{R}$, so we are reduced to prove that $(0,1)$ is not in bijection with $\mathbb{N}$. Assume by contradiction that there is such a bijection. Then, if we express real numbers through their decimal representation, we have an enumeration

$$\begin{array}{c} 0.h_1^1 h_2^1 h_3^1 \dots \\ 0.h_1^2 h_2^2 h_3^2 \dots \\ 0.h_1^3 h_2^3 h_3^3 \dots \\ \vdots \end{array}$$

Now define a new real number $0.k_1k_2k_3\dots$ where $k_n = 1$ whenever h_n^n is even and $k_n = 2$ whenever h_n^n is odd. This number can't be part of this enumeration by construction, so the claimed bijection is not surjective, which is a contradiction.

Week 4

Exercise Characterize all well-ordered sets A such that A^{op} is also well-ordered.

Solution These sets are precisely the finite ones. Indeed, if A is finite then each of its subsets has a maximum element, which becomes a minimum in A^{op}. On the other hand, if A is infinite, then we know that $|\omega| \leq |A|$, so that ω is an initial segment of A. Now take a non-empty subset of ω, and therefore of A, which has no maximum. Its corresponding subset in $\omega^{op} \subseteq A^{op}$ has no minimum.

Exercise Give an example of a well-ordering on the set $\omega \times \omega$.

Solution A possible solution is the lexicographic ordering: for $(a, b), (a', b') \in \omega \times \omega$, set

$$(a, b) < (a', b') \text{ if either } a < a' \text{ or } a = a' \text{ and } b < b'.$$

It is immediate to check that it is a total order. Moreover, suppose $S \subseteq \omega \times \omega$ is non-empty. Then take

$$a_0 := \min\{a \in \omega | \exists b \text{ such that } (a, b) \in S\}$$

which exists because ω is well-ordered, and

$$b_0 := \min\{b \in \omega | (a_0, b) \in S\}$$

which exists for the same reason. Then (a_0, b_0) is the minimum of S.

Another possible well order is:

$$(a, b) < (a', b') \text{ if either } a + b < a' + b' \text{ or } a + b = a' + b' \text{ and } a < a'.$$

$$\begin{tikzcd}
(0,0) \arrow[r] & (0,1) \arrow[dl] & (0,2) \arrow[dl] & (0,3) \arrow[dl] & \cdots \\
(1,0) \arrow[urr] & (1,1) \arrow[dl] & (1,2) \arrow[dl] & (1,3) & \cdots \\
(2,0) \arrow[uurrr] & (2,1) \arrow[dl] & (2,2) & (2,3) & \cdots \\
(3,0) \arrow[uuurrrr] & (3,1) & (3,2) & (3,3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{tikzcd}$$

This is a total order because ω is, and $a + b, a' + b' \in \omega$.
Moreover, given a non-empty subset $S \subseteq \omega \times \omega$, define

$$n_0 := \min\{n \in \omega | \exists a \exists b \text{ such that } (a,b) \in S \text{ and } a+b=n\}$$

which exists because ω is well-ordered. Then define

$$a_0 := \min\{a \in \omega | \exists b \text{ such that } (a_0,b) \in S \text{ and } a+b=n_0\}$$

which again exists for the same reason. Then $(a_0, n_0 - a_0)$ is the minimum of S.

We can give yet another solution: calling $c = \max\{a,b\}$ and $c' = \{a',b'\}$,

$$(a,b) < (a',b') \text{ if } c < c' \text{ or } c = c' \text{ and } a < a' \text{ or } c = c',\ a = a' \text{ and } b < b'.$$

In other words, it restricts to the lexicographic order whenever the maxima coincide.

$$\begin{tikzcd}
(0,0) \arrow[r] & (0,1) \arrow[dl] & (0,2) \arrow[d] & (0,3) \arrow[d] & \dots \\
(1,0) \arrow[r] & (1,1) \arrow[ur] & (1,2) \arrow[dll] & (1,3) \arrow[d] & \dots \\
(2,0) \arrow[r] & (2,1) \arrow[r] & (2,2) \arrow[uur] & (2,3) \arrow[dlll] & \dots \\
(3,0) \arrow[r] & (3,1) \arrow[r] & (3,2) \arrow[r] & (3,3) \arrow[uuur] & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{tikzcd}$$

This is a linear order because ω is linearly ordered and $c, c' \in \omega$.
To see that it is a well-ordering, let us consider a non-empty subset $S \subseteq \omega \times \omega$, and define

$$\begin{gathered}
c_0 := \min\{c \in \omega | \exists (a,b) \in S \text{ such that } \max\{a,b\} = c\} \\
a_0 := \min\{a \in \omega | \exists b \text{ such that } (a,b) \in S \text{ and } \max\{a,b\} = c_0\} \\
b_0 := \min\{b \in \omega | (a_0, b) \in S\}.
\end{gathered}$$

Then the pair (a_0, b_0) is the minimum of S.

Exercise Prove that the set of algebraic numbers is countable.

Solution Remember that a real number r is called algebraic if there is a polynomial $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ with integers coefficients such that $a_0 + a_1 r \ldots + a_n r^n = 0$.
The set of polynomials of degree n is in bijection with $\mathbb{Z}^n$, because every polynomial is determined by its coefficients. Now, we know from algebra that a polynomial of degree n has at most n roots, so the set of real numbers that are roots of polynomials of degree n is contained in $n \times$

$\mathbb{Z}^n$. As a consequence, the set $\overline{\mathbb{Q}}$ of all algebraic numbers is contained in $\bigcup_{n\in\mathbb{N}} n \times \mathbb{Z}^n$.
Now observe that $\mathbb{Z}$ is countable, as we already know, and a finite product of countable sets is easily shown to be countable, so $\mathbb{Z}^n$ is countable as well. A fortiori, we know that $n \times \mathbb{Z}^n$ is countable, because it is a finite product of sets that are at most countable. Finally, a countable union of countable sets is countable, because $\bigcup_{n\in\mathbb{N}} \mathbb{N} \times \{n\} = \mathbb{N} \times \mathbb{N}$. It follows that $\bigcup_{n\in\mathbb{N}} n \times \mathbb{Z}^n$ is countable, so that there is an injective function

$$\overline{\mathbb{Q}} \to \bigcup_{n\in\mathbb{N}} n \times \mathbb{Z}^n \cong \mathbb{N}$$

and $|\overline{\mathbb{Q}}| \leq |\mathbb{N}|$.
For the other direction, observe that every natural number n is algebraic, just taking the polynomial $-n + x$, so the inclusion $\mathbb{N} \subseteq \overline{\mathbb{Q}}$ implies that $|\mathbb{N}| \leq |\overline{\mathbb{Q}}|$.

Axiom of choice The axiom of choice states that for every family of sets $(X_i)_{i\in I}$ there is a function $c : I \to \bigcup_{i\in I} X_i$ such that $\forall i \in I$ we have that $c(i) \in X_i$.

Well-ordering principle The well-ordering principle states that for every set X there can be defined a well-ordering on X.

Exercise Show that the well-ordering principle implies the axiom of choice.

Solution Assume the well-ordering principle. Then make every X_i into a well-ordered set, and define the function

$$c : i \mapsto \min X_i.$$

Remark In fact, the converse implication is also true, that is, assuming the axiom of choice we can define a well-ordering on every set, but the proof of this is much more involved.

Week 5

Exercise Show that $\omega \cdot 2 \ncong \omega$

Solution If there was an isomorphism of ordered sets, each subset of $\omega \cdot 2$ would correspond uniquely to a subset of ω with the same order-theoretic properties. But $\omega \cdot 2$ has an infinite subset with a maximum, namely $\omega + 1$, which ω doesn't have. Therefore they can't be isomorphic.

Definition We denote $\omega_0 := \omega$. Then inductively for every ordinal α we denote as $\omega_{\alpha+1}$ the smallest ordinal number whose cardinality is bigger than $|\omega_\alpha|$. For example, the symbol ω_1 will denote the smallest uncountable ordinal.

Exercise Find, in the class of ordinal numbers:

1. The third smallest infinite ordinal number;
2. some ordinal α such that $\omega^2 < \alpha < \omega^3$;
3. the largest countable ordinal number;

4. the ω-th uncountable ordinal number.

Solution We separate the four requests:

1. The smallest infinite ordinal number is ω, so we only have to take its successor twice, in the class of ordinals. We obtain that $\omega + 2$ is what we are looking for.
2. Any ordinal of the form $\omega^2 \cdot n$ with n finite, or $\omega^2 + \beta$, with $\beta < \omega^3$, satisfies the requirement.
3. No such ordinal exists. Indeed, for each countable ordinal α we have $\alpha < \alpha + 1$, and $\alpha + 1$ is clearly also countable.
4. By ω-th, we mean that the linearly ordered set of all uncountable ordinals smaller than the one we wish to find is isomorphic to ω. Now, since the smallest uncontable ordinal is ω_1, the second smallest uncountable ordinal will be $\omega_1 + 1$, so by induction we obtain that the $(n+1)$-th smallest uncountable ordinal is $\omega_1 + n$. This implies that the ω-th smallest will be $\omega_1 + \omega$.

Exercise Write the following ordinals in the simplest possible way: $1^\omega, 2^\omega, \ldots, n^\omega, \ldots, \omega^\omega$. Determine which ones, if any, are countable.

Solution By definition, 1^ω is the supremum of the set $\{1^m | m < \omega\}$, which is constant on 1. Therefore, $1^\omega = 1$.

For each $1 < n < \omega$, the set n^ω is the supremum of the set $\{n^m | m < \omega\}$. All the elements of this set are finite ordinals, and moreover they are unbounded, because for every $m < \omega$ we have $m < n^m$. Therefore, n^ω is equal to ω itself.

ω^ω is the supremum of the set $\{\omega^n | n < \omega\}$. Since all these ordinals are distinct, there is no simpler way of writing ω^ω. Moreover, it is a countable union of countable sets, therefore it is itself countable.

Observation The examples above are instances of one important fact: the arithmetic of cardinals does not correspond with the arithmetic of ordinals. For example, addition and multiplication are commutative in the former, but not in the latter. Moreover, an operation like α^β can have very different results depending on whether we regard α and β as cardinals or as ordinals. However, it is still true that $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$ and $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$, although a little more complicated to prove (it will be done later on in the course).

Exercise Show that the equality $\alpha \cdot \omega_1 = \omega_1$ holds for each ordinal $1 \leq \alpha < \omega_1$.

Solution $\alpha \cdot \omega_1$ is the supremum of the set $\{\alpha \cdot \beta | \beta \text{ is countable }\}$. Each element $\alpha \cdot \beta$ is countable because it is a product of countable sets, so we have $\alpha \cdot \beta < \omega_1$, which implies $\alpha \cdot \omega_1 \leq \omega_1$.
On the other hand, for each countable ordinal β we have that $\beta \leq \alpha \cdot \beta < \alpha \cdot \omega_1$, so that $\alpha \cdot \omega_1$ is bigger than all countable ordinals, and therefore $\omega_1 \leq \alpha \cdot \omega_1$.
We have proven both the desired inequalities, so $\omega_1 = \alpha \cdot \omega_1$.

Week 6

Exercise We saw in an earlier exercise that the cardinals $\aleph_0, 2\cdot\aleph_0, \aleph_0\cdot 2, 2^{\aleph_0}, \aleph_0^{\aleph_0}, c^{\aleph_0}, \aleph_0^c, \aleph_0\cdot\aleph_0$ are ordered as

$$\aleph_0 = 2\cdot\aleph_0 = \aleph_0\cdot 2 = \aleph_0\cdot\aleph_0 < 2^{\aleph_0} = \aleph_0^{\aleph_0} = c^{\aleph_0} < \aleph_0^c.$$

We want to do an analogous exercise, but regarding the given numbers as ordinals instead. In other words, find the increasing ordinal order of $\omega, 2\cdot\omega, \omega\cdot 2, 2^\omega, \omega^\omega, \omega_1^\omega, \omega^{\omega_1}, \omega\cdot\omega$.

Solution First, recall from the lecture that $2\cdot\omega = \omega$. Moreover, we have seen above that $\omega\cdot 2 \not\cong \omega$, so we obviously conclude that $\omega < \omega\cdot 2$. In another exercise, we have seen that $2^\omega = \omega$, while ω^ω is bigger than all ω^n's and therefore all $\omega\cdot n$'s, but it is still countable, and it behaves very differently in regard to the respective order relations.
Next we compute

$$\omega^{\omega_1} = (2^\omega)^{\omega_1} = 2^{\omega\cdot\omega_1} = 2^{\omega_1} = \omega_1.$$

As for ω_1^ω, this is the supremum of the set $\{\omega_1^n | n < \omega\}$, all of whose elements are strictly bigger than ω_1, so $\omega_1^\omega > \omega_1$.
Finally, we know that $\omega\cdot\omega$ is countable, and it is obviously bigger than $\omega\cdot 2$ but smaller than ω^ω.
In conclusion, the order of the given ordinals is

$$\omega = 2\cdot\omega = 2^\omega < \omega\cdot 2 < \omega\cdot\omega < \omega^\omega < \omega^{\omega_1} < \omega_1^\omega.$$

Exercise Find the smallest ordinal number α such that the power α^ω is

1. a finite ordinal number;
2. a countable ordinal number;
3. an uncountable ordinal number.

Solution The solution to (1) is immediate, because $0^\omega = 0$, and 0 is the smallest ordinal number.
For (2), observe that $1^\omega = 1$ and $2^\omega = \omega$. Since $2 = 1^+$, our solution is exactly 2.
For (3), observe that for every countable ordinal number β we have $\beta^\omega = \sup\{\beta^n | n \in \mathbb{N}\}$. Now, β^n is the product of a finite number of countable sets, therefore it is countable itself. It follows that β^ω, being a countable union of countable sets, is itself countable. So the ordinal we are looking for must be uncountable, therefore at least ω_1. Since ω_1^ω is uncountable, our solution is precisely ω_1.

Exercise Define ω_ω as the smallest ordinal whose cardinality is strictly bigger than the cardinality of all ω_n's. In other words,

$$\omega_\omega := \min\{\alpha, \forall n \in \mathbb{N} |\omega_n| < |\alpha|\}.$$

Show that $\omega_\omega = \sup_n \omega_n$.

Solution We will show both inequalities. First, observe that $|\omega_n| < |\omega_\omega|$ implies $\omega_n < \omega_\omega$ for each $n \in \mathbb{N}$, so that, by definition of supremum, we obtain

$$\sup_n \omega_n \leq \omega_\omega.$$

Conversely, since for each n the inequality $\omega_n < \sup_n \omega_n$ holds, it must be true that $|\omega_n| \leq |\sup_n \omega_n|$ (with the weak inequality sign, for the moment). Thus we obtain

$$|\omega_n| < |\omega_{n+1}| \leq |\sup_n \omega_n|$$

now with a strict inequality sign. By definition of ω_ω, this means that

$$\omega_\omega \leq \sup_n \omega_n.$$

Definition Remember that for every ordinal number $\alpha > 0$ there are non-null natural numbers $k, m_0, \ldots, m_k$ and ordinal numbers $\gamma_0 > \ldots > \gamma_k$ such that

$$\alpha = \omega^{\gamma_0} \cdot m_0 + \ldots + \omega^{\gamma_k} \cdot m_k.$$

This expression is known as the Cantor normal form of α.

Exercise Find the Cantor normal form of ω_n, or each n, and ω_ω.

Solution Following the first steps of the proof of the existence theorem for Cantor normal forms, we start by looking at the set

$$X := \{\gamma | \omega^\gamma \leq \omega_n\}$$

and then we study the ordinal $\omega^{\sup X}$.

Now, if $n = 0$, then $X = \{0, 1\}$, so that the above ordinal becomes $\omega^1 = \omega = \omega_0$, and we are done.
If $0 < n < \omega$, then X is the set of all ordinals of cardinality strictly smaller than ω_n, so that $\sup X = \omega_n$. Thus we obtain that $\omega^{\omega_n} \leq \omega_n$. We only need to show the other inequality. For that, observe that by previous exercises we know that $2^{\omega_n} = \omega_n$, so compute

$$\omega_n = 2^{\omega_n} \leq \omega^{\omega_n}.$$

Finally, turning to ω_ω, our X will be the set of all ordinals of cardinality strictly smaller than ω_ω, therefore the same argument shows that the Cantor normal form turns out to be ω^{ω_ω}.

Remark The same line of reasoning proves in fact a more general result: if α is an ordinal which is the smallest with its cardinality, then its Cantor normal form is ω^α.

Proposition The Cantor normal form is unique.

Proof Consider an ordinal α, and take two Cantor normal forms for it

$$\omega^{\gamma_0} \cdot m_0 + \ldots + \omega^{\gamma_k} \cdot m_k$$

and

$$\omega^{\delta_0} \cdot n_0 + \ldots + \omega^{\delta_l} \cdot n_l$$

assuming without loss of generality that $k \leq l$.
Since $\gamma_0 > \ldots > \gamma_k$ and $\delta_0 > \ldots > \delta_l$, if $\gamma_0 < \delta_0$ we would have that the first normal form is smaller than the second, which is absurd. Therefore, $\gamma_0 = \delta_0$. For the same reason, we must have $m_0 = n_0$. Inductively, we obtain that for $0 \leq i \leq k$ we have $\gamma_i = \delta_i$ and $m_i = n_i$.
It only remains to prove that $k = l$. Suppose by contradiction that $k < l$. Then the remaining terms of the second normal form sum up to an ordinal $\beta \neq 0$. Therefore, we obtain that first normal form gives α, while the second gives $\alpha + \beta > \alpha$, which is absurd. This concludes the proof.

Week 7

Exercise Order the following ordinals increasingly: $\omega + 1 + \omega, \omega \cdot (\omega + 1), (1 + \omega) \cdot \omega, (1 + \omega) \cdot (\omega + 1), \omega + 1 + \omega^2 + \omega$.

Solution Remember that sum of ordinal numbers is associative, so the first is equal to

$$\omega + (1 + \omega) = \omega + \omega.$$

Also remember that multiplication distributes with respect to addition, so we have

$$\omega \cdot (\omega + 1) = \omega \cdot \omega + \omega = \omega^2 + \omega.$$

Next, we have again $(1 + \omega) \cdot \omega = \omega \cdot \omega = \omega^2$.
Use the same principles to compute

$$(1 + \omega) \cdot (\omega + 1) = \omega \cdot (\omega + 1) = \omega^2 + \omega.$$

Finally, we have

$$\omega + (1 + \omega^2 + \omega) = \omega + (\omega^2 + \omega) = (\omega + \omega^2) + \omega = \omega \cdot (1 + \omega) + \omega = \omega \cdot \omega + \omega = \omega^2 + \omega.$$

These ordinals are then ordered as

$$\omega + 1 + \omega < (1 + \omega) \cdot \omega < \omega \cdot (\omega + 1) = (1 + \omega) \cdot (\omega + 1) = \omega + 1 + \omega^2 + \omega.$$

Exercise Decide whether $\omega^\omega = \omega$ or $\omega^{(\omega^\omega)} = \omega^\omega$.

Solution Both are false. The proof of the first is analogous to the proof of $\omega \cdot 2 \neq \omega$, which extends to all $\omega \cdot n$'s and all ω^n's.
As for the second, consider in both sets the subset of all elements γ such that the set $\{\delta | \delta < \gamma\}$ has no maximum. For the left-hand side, this is isomorphic (as an ordered set) to ω^ω, for the right-hand side it is isomorphic to ω, so the result follows from the first part.

Exercise Define ϵ_0 to be the smallest ordinal α such that $\omega^\alpha = \alpha$. Prove that $\epsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$.

Solution The ordinal $\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}$ is defined to be the supremum of the sequence of ordinals defined by $\alpha_0 = \omega$ and for each $n \in \omega$, $\alpha_{n+1} = \omega^{\alpha_n}$. First, observe that, this limit obviously has the required property. Therefore, by minimality of ϵ_0, we have

$$\epsilon_0 \leq \omega^{\omega^{\cdot^{\cdot^{\cdot}}}}.$$

To prove the opposite inequality, let us start by observing that ϵ_0 cannot be finite, so $\alpha_0 = \omega \leq \epsilon_0$. Recursively, exponentiating ω with the n-th obtained inequality, we get $\alpha_{n+1} = \omega^{\alpha_n} \leq \omega^{\epsilon_0} = \epsilon_0$, so $\alpha_n \leq \epsilon_0$, for each α_n of the sequence defined above. Therefore, by the property of the supremum, we obtain

$$\omega^{\omega^{\cdot^{\cdot^{\cdot}}}} \leq \epsilon_0.$$

Exercise Show that the axiom of choice is equivalent to the statement: every surjective function $f : X \to Y$ has a right inverse.

Solution Assuming the axiom of choice, then we can choose a point $x_y \in X_y$ for every fiber of f, and define a function by $y \to x_y$. This is clearly a right inverse to f.
Conversely, consider a family of sets $(X_i)_{i \in I}$, and the projection function $\coprod_{i \in I} X_i \to I$. This is surjective, then by assumption it has a right inverse $c : I \to \coprod_{i \in I} X_i$. Therefore, for every $i \in I$ we have $c(i) \in p_i = X_i$, so this is a choice function.

Curiosities The axiom of choice has been proven to be independent of ZF set theory. Therefore, both $ZF + AC$ and $ZF + \neg AC$ are consistent theories (provided ZF is). Both AC and its negation $\neg AC$ have weird and counterintuitive consequences, the most famous of which, assuming AC is Banach-Tarski paradox: given a 3-dimensional full ball, there is a way of cutting it into a finite number of pieces, and then reassemble these to form two spheres identical in size to the original one.
Another statement which is equivalent to AC is:

– Every vector space has a basis.

On the contrary, assuming $\neg AC$, we have these fun facts:

- There is an infinite set X with no injection $\mathbb{N} \to X$;
- there is a set X and a partition of X such that there are more elements in the partition than elements in the set;
- there are families of non-empty sets whose Cartesian product is empty;
- there are vector spaces with no basis;
- it is possible to express real numbers as a countable union of countable sets;
- it is furthermore possible to express real numbers as a union of two subsets of strictly smaller cardinality.

Week 8

Definition The cofinality of a cardinal κ is the minimum size of a set of cardinals smaller than κ whose sum is κ, i.e.

$$\mathrm{cf}(\kappa) = \min\{|I|, \kappa = \textstyle\sum_{i \in I} \lambda_i \text{ with } \forall i \in I, \lambda_i < \kappa\}.$$

Definition Also remember, a cardinal κ is called regular if for every family of cardinals $(\lambda_i)_{i \in I}$ such that $\lambda_i < \kappa$ and $|I| < \kappa$, then

$$\textstyle\sum_{i \in I} \lambda_i < \kappa.$$

Remark Obviously, for every cardinal we have $\mathrm{cf}(\kappa) \leq \kappa$, because $\sum_\kappa 1 = \kappa$. The statement that κ is a regular cardinal is then equivalent to the statement $\mathrm{cf}(\kappa) = \kappa$.

Definition The sequence of $\aleph$ numbers is recursively defined on ordinals:

- $\aleph_0 = \omega$;
- for an ordinal α, $\aleph_{\alpha+1} = \aleph_\alpha^+$;
- for a limit ordinal γ, $\aleph_\gamma = \bigcup_{\beta < \gamma} \aleph_\beta$.

Definition The sequence of $\beth$ numbers is recursively defined on ordinals:

- $\beth_0 = \omega$;
- for an ordinal α, $\beth_{\alpha+1} = 2^{\beth_\alpha}$;
- for a limit ordinal γ, $\beth_\gamma = \bigcup_{\beta < \gamma} \beth_\beta$.

Remark Cantor's theorem states that we always have $\kappa < 2^\kappa$. Therefore, we also always have $\aleph_\alpha \leq \beth_\alpha$.

Remark With this notation, the continuum hypothesis states that $\aleph_1 = \beth_1$. In particular, it is consistent with ZFC that $\beth_1 = \aleph_0^+$. It is also consistent that $\beth_1 = \aleph_0^{++}$. In fact, for any finite number n, the statement

$$\beth_1 = \aleph_0^{+^n} = \aleph_n$$

is consistent with ZFC.
Therefore, the statement that for every ordinal α and every natural number n the equality

$$\beth_\alpha = \aleph_{\alpha+n}$$

holds is consistent with ZFC.

Remark Even more, given any cardinal κ, the following statement is consistent with ZFC: there is a possibly infinite ordinal λ such that $2^\kappa = \kappa^{+^\lambda}$.
We can choose such an ordinal λ for each κ with some degree of liberty. However, it has been proven that if we want to choose a single λ such that the equality $2^\kappa = \kappa^{+^\lambda}$ is true simultaneously for every cardinal κ, then λ needs to be finite.

Remark The generalized continuum hypothesis can be reformulated as

$$\forall \alpha, \aleph_\alpha = \beth_\alpha.$$

Definition A cardinal κ is called a strong limit if whenever $\lambda < \kappa$ then $2^\lambda < \kappa$.
It is called a weak limit if whenever $\lambda < \kappa$ then $\lambda^+ < \kappa$.
Furthermore, κ is strongly inaccessible if it is regular and a strong limit, it is weakly inaccessible if it is regular and a weak limit.

Remark Every, strong limit is clearly also a weak limit, and every strong inaccessible is a weak inaccessible. Under GCH, the definitions of strong limit and weak limit are equivalent, and the definitions of strongly inaccessible and weakly inaccessible are likewise equivalent.

Exercise Show that, for each finite n, the cardinal $\aleph_n$ is regular but, if $n \geq 1$, it is not a strong or weak limit. Further show that $\aleph_\omega$ is a weak limit but not regular. Decide if $\aleph_\omega$ is a strong limit.

Solution We know that a finite sum of finite numbers is finite, which means that $\aleph_0$ is regular. For any n, assume that we have a family of cardinals $(\lambda_i)_{i\in I}$ with $|I| \leq \aleph_n$ and $\lambda_i \leq \aleph_n$, therefore we can compute

$$\textstyle\sum_{i\in I} \lambda_i \leq \sum_{i\in I} \aleph_n = |I| \times \aleph_n \leq \aleph_n \cdot \aleph_n = \aleph_n$$

which proves that $\aleph_{n+1}$ is regular.
For each n we have that $\aleph_n < \aleph_{n+1}$ but $2^{\aleph_n} \geq \aleph_n^+ = \aleph_{n+1}$ so $\aleph_{n+1}$ is not a weak or a strong limit.
$\aleph_\omega$ is not regular because by its very definition $\sum_{n\in\omega} \aleph_n = \aleph_\omega$.
Moreover, it is a weak limit because every cardinal smaller than $\aleph_\omega$ is either finite or of the form $\aleph_n$. It is obvious that both $n+1 < \aleph_\omega$ and that $\aleph_n^+ = \aleph_{n+1} < \aleph_\omega$.
In ZFC, it is not possible to prove that $\aleph_\omega$ either is a strong limit or is not such. Under GCH, for instance, it is obviously a strong limit because strong limits coincide with weak limits. On the other hand, it is consistent with ZFC that there is an n and and infinite λ such that $2^{\aleph_n} = \aleph_n^{+^\lambda}$. In this case, $\aleph_n < \aleph_\omega$, but $2^{\aleph_n} \geq \aleph_\omega$.

Remark By transfinite induction, we can prove that it is always the case that $\kappa \leq \aleph_\kappa$, for an arbitrary cardinal κ.

Remark Whenever α is an initial ordinal (least with its cardinality) then we have $\mathrm{cf}(\aleph_\alpha) = \alpha$.

Exercise Show that an uncountable cardinal κ is weakly inaccessible if and only if $\kappa = \aleph_\kappa$.

Solution Assume that $\kappa = \aleph_\kappa$. Then we have $\mathrm{cf}(\kappa) = \mathrm{cf}(\aleph_\kappa) = \kappa$ so that κ is regular. Take now a cardinal $\mu < \kappa$, therefore $\mu < \aleph_\kappa$, so that there is an ordinal $\alpha < \kappa$ such that $\mu < \aleph_\alpha$. Therefore $\mu^+ < \aleph_{\alpha+1} < \aleph_\kappa = \kappa$.
Conversely, if κ is weakly inaccessible. Since $\kappa \leq \aleph_\kappa$, we need to prove the opposite inequality. We have $\aleph_0 < \kappa$ by hypothesis. Since κ is a weak limit, for each ordinal α we also have that

$$\aleph_\alpha < \kappa \Rightarrow \aleph_{\alpha+1} < \kappa.$$

Finally, for a limit ordinal $\lambda < \kappa$, by the previous step and regularity of κ we deduce that

$$\aleph_\lambda < \kappa.$$

By definition of $\aleph$ number associated to a limit ordinal, this implies that $\aleph_\kappa \leq \kappa$.

Remark The same reasoning yields that an uncountable cardinal κ is strongly inaccessible if and only if $\kappa = \beth_\kappa$.

Remark There is a function, called the $\aleph$ function, from the class of cardinals to itself, defined by $\kappa \mapsto \aleph_\kappa$. A weakly inaccessible cardinal can be defined as a fixed point of the $\aleph$ function.
Similarly we can define strongly inaccessible cardinals as fixed points of an analogously defined $\beth$ function.

Remark The existence of one inaccessible cardinal is independent of ZFC. Moreover, the existence of $\lambda + 1$ inaccessible cardinals is independent of ZFC + "there are λ inaccessible cardinals". Finally, the existence of a proper class of inaccessible cardinals is independent of ZFC + "there are λ inaccessible cardinals" for every λ.

Week 9

Exercise Show that, for a set x, we have $\mathrm{rk}(x) = \sup\{\mathrm{rk}(z) + 1 | z \in x\}$.

Solution Let $\alpha = \sup\{\mathrm{rk}(z) + 1 | z \in x\}$. This means in particular that for every $z \in x$ we have $z \subseteq V_\beta$ for some β such that $\beta + 1 \leq \alpha$, which implies $z \in V_\alpha$. As a consequence, $x \subseteq V_\alpha$, which means that $\mathrm{rk}(x) \leq \alpha$. We want to show that this is an equality. Suppose by contradition that $\mathrm{rk}(x) = \gamma < \alpha$. In other words, $x \subseteq V_\gamma$. Then for each element $z \in x$ we have $z \in V_\gamma$, which contradicts the minimality of α. This proves that $\mathrm{rk}(x) = \alpha$, as desired.

Remark Another way to express this is: if $\exists \max\{\mathrm{rk}(z)|z \in x\}$ then $\mathrm{rk}(x) = \max\{\mathrm{rk}(z)|z \in x\} + 1$.
If that maximum doesn't exist, then $\mathrm{rk}(x) = \sup\{\mathrm{rk}(z)|z \in x\}$.

Exercise Prove by induction that $|V_{\omega+\alpha}| = \beth_\alpha$.

Solution For $\alpha = 0$, notice that V_ω is a countable union of finite sets, which is bigger than any finite set, therefore $|V_\omega| = \omega = \beth_0$.

Now consider the isolated step, and assume by induction that $|V_{\omega+\alpha}| = \beth_\alpha$. Then $|V_{\omega+\alpha+1}| = |\mathcal{P}(V_{\omega+\alpha})| = 2^{|V_{\omega+\alpha}|} = 2^{\beth_\alpha} = \beth_{\alpha+1}$.

Finally, if α is a limit ordinal, then

$$|V_{\omega+\alpha}| = |\bigcup_{\beta<\alpha} V_{\omega+\beta}| = \bigcup_{\beta<\alpha} |V_{\omega+\beta}| = \bigcup_{\beta<\alpha} \beth_\beta = \beth_\alpha.$$

Exercise Conclude that if κ is an inaccessible cardinal, then $|V_\kappa| = \kappa$.

Solution We can regard a cardinal as an initial ordinal. Since κ is uncountable, we have $\omega + \kappa = \kappa$, thus the exercise above says

$$|V_\kappa| = |V_{\omega+\kappa}| = \beth_\kappa$$

but we know from last week that $\beth_\kappa = \kappa$, so we are done.

Exercise Show that, if x and y have rank $\leq \alpha$ then $\{x, y\}$, $x \cup y$, $\bigcup x$, $\mathcal{P}(x)$, (x, y) and x^y have rank $< \alpha + \omega$.

Solution $\{x, y\}$ has rank $\leq \alpha + 1$ by the exercise above.

Next, for each element $z \in x$ or $z \in y$ we have $z \in V_\beta$ for some $\beta \leq \alpha$, therefore, again by the same exercise above, we conclude $\mathrm{rk}(x \cup y) \leq \alpha + 1$.

By $\bigcup x$ we mean the union of all sets y such that $y \in x$. Since $\mathrm{rk}(x) \leq \alpha$, we also have a fortiori $\mathrm{rk}(y) \leq \alpha$ and, for each element $z \in y$, $\mathrm{rk}(z) \leq \alpha$. Therefore $\mathrm{rk}(\bigcup x) \leq \alpha$. In fact, we even have $\mathrm{rk}(\bigcup x) < \mathrm{rk}(x)$.

If $\mathrm{rk}(x) = \beta \leq \alpha$, then for each $z \in x$ we have $\mathrm{rk}(z) \leq \beta$, so for each subset $y \subseteq x$, by the exercise above we deduce $\mathrm{rk}(y) \leq \beta + 1$. Thus, $\mathrm{rk}(\mathcal{P}(x)) \leq \beta + 2 \leq \alpha + 2$.

We use the definition $(x, y) = \{x, \{x, y\}\}$. By the above reasoning, $\mathrm{rk}(\{x, y\}) \leq \alpha + 1$, so by the exercise above we have $\mathrm{rk}((x, y)) \leq \max\{\alpha + 1, \alpha + 2\} = \alpha + 2$.

A function $y \to x$ is defined as a specific subset of $y \times x$. Now, if y and x have rank $\leq \alpha$, every element of them also has rank $\leq \alpha$, therefore by the previous part a pair (y_0, x_0) where $y_0 \in y$ and $x_0 \in y$ has rank $\leq \alpha + 2$. The above exercise implies that a function $y \to x$, when regarded as a set, has rank $\leq \alpha + 3$. The set x^y of all functions $y \to x$ now has rank $\leq \alpha + 4$.

Week 10

Definition The axiom schema of replacement says that for each formula ϕ containing the variables $x, t_1, \ldots, t_n, y$ there is an axiom as follows

$$\forall x \forall y \forall z \left(\phi\left(x, t_1, \ldots, t_n, y\right) \wedge \phi\left(x, t_1, \ldots, t_n, z\right) \rightarrow y = z\right) \rightarrow \\ \forall X \exists Y \forall y \left(y \in Y \leftrightarrow \exists x \left(x \in X \wedge \phi\left(x, t_1, \ldots, t_n, y\right)\right)\right)$$

Intuition We should think of ϕ as a relation which depends on parameters $t_1, \ldots, t_n$, so for the sake of intuition we now abbreviate $\phi(x, t_1, \ldots, t_n, y)$ by xRy. Now what the first part of the above axiom is saying is that R is a binary relation such that to each x there is at most one y which is in relation with it. The second half of it says that if X is a set, then the elements y that are associated to some element $x \in X$ form themselves a set Y

One more layer of simplification is the following: if R is a function (wherever it is defined) then the images of elements that range over a set X form themselves a set Y.

The non-triviality of this axiom lies in the fact that, if R is a function, its codomain is a priori the class of all sets, which is not a set. What the axiom says is, in these terms, that the image of a set along a class function is a set.

Definition A Grothendieck universe is a set $\mathcal{U}$ satisfying the following axioms:

1. $X \in \mathcal{U}$ and $Y \in X$ implies $Y \in \mathcal{U}$;
2. If $X \in \mathcal{U}$, then also $\mathcal{P}(X) \in \mathcal{U}$;
3. If $I \in \mathcal{U}$ and $(X_i)_{i \in I}$ is an I-indexed family of sets in $\mathcal{U}$, then $\bigcup_{i \in I} X_i \in \mathcal{U}$;
4. $\mathbb{N} \in \mathcal{U}$.

Remark If $\mathcal{U}$ is a Grothendieck universe and $X \in \mathcal{U}$, then $|X| < |\mathcal{U}|$. To see this, simply use axiom 2 to see that $\mathcal{P}(X) \in \mathcal{U}$, then observe that by axiom 1 $\mathcal{P}(X) \subseteq \mathcal{U}$, therefore $|X| < |\mathcal{P}(X)| \leq |\mathcal{U}|$.

Exercise Show that if $X \in \mathcal{U}$ and $Y \subseteq X$ then $Y \in \mathcal{U}$.

Solution We have $Y \in \mathcal{P}(X) \in \mathcal{U}$ by axiom 2. Then we conclude by axiom 2.

Exercise Show that if $\mu < |\mathcal{U}|$, then there is a set $X \in \mathcal{U}$ such that $|X| = \mu$.

Solution By the previous exercise and the fact that $\mu \leq \beth_\mu$, it suffices to show that there is a set $Y \in \mathcal{U}$ such that $|Y| = \beth_\mu$. Let us proceed by induction on μ.
If $\mu = 0$, then the result follows by axiom 4.
Assume by induction that we have proven the result up to α, then there is a set $Y_\alpha \in \mathcal{U}$ with cardinality $\beth_\alpha$. By axiom 2, its power set belongs to $\mathcal{U}$ and, moreover, it has cardinality $\beth_{\alpha+1}$ by definition.
Now assume that λ is a limit ordinal, and that for each $\alpha < \lambda$ there is a set

$Y_\alpha \in \mathcal{U}$ with cardinality $\beth_\alpha$. Assume that $\lambda < \beth_\lambda$ (the proof when $\lambda = \beth_\lambda$ is much more complicated), so there is some $\alpha < \lambda$ such that $\lambda < \beth_\alpha$. By inductive hypothesis, there is a set $Y_\alpha \in \mathcal{U}$ with cardinality $\beth_\alpha$, so by the previous exercise there must be also a set $I \in \mathcal{U}$ with cardinality λ, so that we can index a family $(Y_i)_{i \in I}$ including all the sets of the form Y_α with $\alpha < \lambda$. Now the union $\bigcup_{i \in I} Y_i$ belongs to $\mathcal{U}$ by axiom 3, and it is equal to $\bigcup_{\alpha < \lambda} Y_\alpha$, which has cardinality $\beth_\lambda$. This concludes the proof.

Exercise Show that, if $\mathcal{U}$ is a Grothendieck universe, then its cardinality is uncountable and strongly inaccessible.

Solution By axiom 4, we have that $\mathbb{N} \in \mathcal{U}$, so the above remark implies that $|\mathbb{N}| < |\mathcal{U}|$, which means that $\mathcal{U}$ is uncountable.
Let $|I| < |\mathcal{U}|$ and $(X_i)_{i \in I}$ a family of sets such that $|X_i| < |\mathcal{U}|$. By the previous exercise, we know that $I, X_i \in \mathcal{U}$, therefore $\bigcup_{i \in I} X_i \in \mathcal{U}$. Again, we conclude by the same remark that the cardinality of this union is strictly smaller than that of $\mathcal{U}$. Thus, $|\mathcal{U}|$ is a regular cardinal.
Finally, if $\mu < |\mathcal{U}|$, choosing a set X such that $|X| = \mu$, we have again by the previous exercise that $X \in \mathcal{U}$, so axiom 2 implies that $\mathcal{P}(X) \in \mathcal{U}$. Once more, we use the same remark to compute $2^\mu = |\mathcal{P}(X)| < |\mathcal{U}|$.

Remark In particular, this implies that the existence of a Grothendieck universe is not provable in ZFC.

Theorem [ZFC] If $\mathcal{U}$ is a Grothendieck universe, then it is a model of ZFC.

Proof This is essentially the same proof seen in the lecture to see that V_κ is a model of ZFC when κ is inaccessible. In fact, we have the following stronger result.

Proposition κ is an inaccessible cardinal if and only if V_κ is a Grothendieck universe.

Proof Observe that axioms 1

Tarski axiom Letting Univ(y) be an abbreviation for all five axioms that say that y is a Grothendieck universe, then Tarski axiom can be formulate as follows

$$\forall x \exists y (x \in y \wedge \mathrm{Univ}(y))$$

Definition Naturally, Tarski axiom is not provable in ZFC. We call TG the theory having all axioms of ZFC + Tarski axiom.

Theorem [TG] ZFC is consistent.

Proof By Tarski axiom, there is a Grothendieck universe $\mathcal{U}$. Then this is a model of ZFC.

Week 11

Definition Recall that a set X is called transitive if $\forall x$ we have $x \in X \Rightarrow x \subset X$.

Remark Observe that the $\subset$ sign in the definition above can never become an $=$ sign, because that would contradict the axiom of regularity.

Definition Also recall that the transitive closure of a set X is the smallest set $TC(X) \supseteq X$ which is transitive. It can be explicitly defined as follows: fix $X_0 = X$ and then inductively

$$X_{n+1} = \bigcup X_n.$$

Then we have $TC(X) = \bigcup_{n \in \mathbb{N}} X_n$.

- We recall one last definition:

Definition A set x is called symmetric is there is a finite set of atoms $N(x) = \{a_1, \ldots, a_n\}$ such that whenever we have a permutation $f : A \to A$ such that $\forall i = 1, \ldots n, f(a_i) = a_i$ then the extension $\hat{f} : W^A \to W^A$ satisfies $\hat{f}(x) = x$.
We say that a set x is hereditarily symmetric if it is symmetric and every element of $TC(x)$ is symmetric.

Remark Observe that the definition of carrier of a set is ambiguous, because it does not fix a single set $N(x)$, rather it is based on the fact that such a set exists. So we will slightly modify this definition for the moment.

Definition For a set $x \in W^A$, define $\mathrm{Carr}(x) \subseteq \mathcal{P}_{fin}(A)$ to be the collection of all carriers of x. They naturally form a poset via the inclusion relation. We will say that a set is symmetric if it satisfies $\mathrm{Carr}(x) \neq \emptyset$.

Exercise Prove the following:

1. Every set $x \in V$ is hereditarily symmetric with carrier $N(x) = \emptyset$;
2. Every atom $a \in A$ is hereditarily symmetric with carrier $N(a) = \{a\}$;

Solution

1. Observe that if $x \in V$ then we also have $TC(x) \in V$. Therefore, this amounts to prove that for every set $x \in V$ and every permutation $f : A \to A$ we have $\hat{f}(x) = x$. Proceed inductively on bounded stages of the universe of discourse V. For $V_0 =$ there is nothing to prove, so we start at height 1. The restriction of f_1 to V_1 is defined as follows: for $x \in V_1$, we necessarily have $x = \emptyset$, therefore $f_1(x) = \{f_0(y) | y \in x\} = \emptyset = x$.
   Now assume that the for all ordinals up to α we have $f_\alpha(x) = x$ whenever $x \in V$. Take $x \in V_{\alpha+1} = \mathcal{P}(V_\alpha)$. For all $y \in x$, we have $y \in V_\alpha$, so by inductive hypothesis $f_\alpha(y) = y$. Therefore, we have, by definition,

   $$f_{\alpha+1}(x) = \{f_\alpha(y) | y \in x\} = \{y | y \in x\} = x.$$

   Finally, if α is a limit ordinal and $x \in V_\alpha$, then $x \in V_\beta$ for some $\beta < \alpha$, so the result is true by directly applying the inductive hypothesis.
2. Since atoms have no elements by definition, we have $TC(a) = a$. Therefore, we only need to show that a is symmetric with carrier $N(a) = \{a\}$. But this is obvious, since whenever $f(a) = a$, then $\hat{f}(a) = a$ by definition.

Exercise Give an example of a set that has two carriers C_1 and C_2 such that $C_1 \cap C_2$ is not a carrier.

Solution For example, consider the set of atoms $A = \{a, b, c, d\}$, and take $x = \{a, b\}$. Now, if $C_1 = \{a, b\}$, then it is obviously a carrier of x. On the other hand, taking $C_2 = \{c, d\}$ also gives a carrier of x, because the only two permutation that fix C_2 are the identity and the transposition (a, b), and in both cases $\hat{f}(\{a, b\}) = \{a, b\}$.
Finally, we have $C_1 \cap C_2 = \emptyset$ and this is not a carrier of x, because for example if f is the cycle (a, b, c, d) then $\hat{f}(x) = \{b, c\} \neq x$.

Exercise Give an example of a set that is symmetric but not hereditarily symmetric.

Solution Let A be an infinite set containing a subset C such that both C and $A \setminus C$ are infinite. Consider now $\mathcal{P}(A) \in W_2^A$. This is clearly symmetric with carrier $\emptyset$, because if $f : A \to A$ is a permutation, then the images along $\hat{f}$ of all subsets of A exhaust all subsets of A.
On the other hand C is an element of $TC(\mathcal{P}(A))$, but this is not symmetric, because both itself and its complement are infinite.

Exercise Give an example of a set that is not symmetric but all its elements are symmetric.

Solution It suffices to take the set C from the exercise above.

Exercise Give an example of a set x such that $TC(x)$ contains all the atoms, every element of x is symmetric but x is not.

Solution Consider an infinite set A, and take as set of atoms the union of two distinct copies $A \coprod A$. So we denote every element either as a_1 or as a_2.
Now, consider the set $x = \{\{a_1, a_2\}\}_{a \in A}$. Every element of x is a finite set of atoms, so it is symmetric having itself as a carrier. Moreover, we have $TC(x) = x \cup (A \coprod A)$, which contains all the atoms. We need to show that x is not symmetric.
Let $C \subseteq A \coprod A$ be any finite subset, and select two elements $a_1, b_1 \notin C$. Define the permutation $f : A \coprod A \to A \coprod A$ as follows: on the first copy of A, it is the identity; on the second copy of it, it is the transposition (a, b). Now we have $\hat{f}(\{b_1, b_2\}) = \{b_1, a_2\}$. This set is an element of $\hat{f}(x)$, but not an element of x, therefore $\hat{f}(x) \neq x$.

Week 12

Exercise Given a filter $\mathcal{F} \subseteq \mathcal{P}(X)$, prove that $\mathcal{F}$ is an ultrafilter if and only if for every finite union $A = \bigcup_{i=1}^{n} A_i \subseteq X$ such that $A \in \mathcal{F}$ then we have $A_i \in \mathcal{F}$ for at least one of the indices i.

Solution Assume that $\mathcal{F}$ is an ultrafilter, and by contradiction assume that there is a finite union $A \in \mathcal{F}$ such that for every index i we have $A_i \notin \mathcal{F}$. By the ultrafilter property, we have $X \setminus A_i \in \mathcal{F}$. Therefore, since elements of a filter are closed under finite intersection, we have $X \setminus A = \bigcap_{i=0}^{n}(X \setminus A_i) \in \mathcal{F}$. Thus, we also have $\emptyset = A \cap (X \setminus A) \in \mathcal{F}$ which is impossible because every ultrafilter is proper.

Conversely, assume that $\mathcal{F}$ is a filter but not an ultrafilter. Then there is a set $A \subseteq X$ such that $A \notin \mathcal{F}$ and $X \setminus A \notin \mathcal{F}$, but obviously we have $A \cup (X \setminus A) = X \in \mathcal{F}$.

Definition The cofinite filter is that defined as $\{A \subseteq X | X \setminus A \text{ is finite }\}$.

Exercise Show that any non-principal ultrafilter contains the cofinite filter.

Solution Let $\mathcal{F}$ be a non-principal ultrafilter, and let $A \subseteq X$ be such that $X \setminus A$ is finite. Since $\mathcal{F}$ is an ultrafilter, it will suffice to show that $X \setminus A \notin \mathcal{F}$. In other words, we need to show that $\mathcal{F}$ does not contain any finite set. Since elements of $\mathcal{F}$ are closed under finite intersection, we are reduced to show that for every $x \in X$, we have $\{x\} \notin \mathcal{F}$. Suppose by contradiction that there is $x \in X$ such that $\{x\} \in \mathcal{F}$. Then every set containing $\{x\}$ is in $\mathcal{F}$, and every set not containing it is not in $\mathcal{F}$, because otherwise we would have $\emptyset \in \mathcal{F}$. In other words, $\mathcal{F}$ is the principal ideal generated by x, which contradicts our hypothesis.

Definition A two-valued measure μ on a set X is called λ-additive if for every set I such that $|I| < \lambda$ and every family $(A_i)_{i \in I}$ of pairwise disjoint subsets of X, we have $\mu(\bigcup_{i \in I} A_i) = \sum_{i \in I} \mu(A_i)$.

Example For a point $x \in X$, we can always define a two-valued λ-additive measure on X given by $\mu(A) = 1$ if and only if $x \in A$.

Definition An uncountable cardinal λ is called measurable if for every set of cardinality λ it is possible to define a non-trivial two-valued λ-additive measure on it.

- The next definitions and propositions give an idea of how large these cardinals are.

Definition Let us give a definition which is recursive on ordinals. We call a cardinal 0-inaccessible if it is inaccessible. For an ordinal $\alpha > 0$, we say that a cardinal κ is α-inaccessible if it is inaccessible and for every ordinal $\beta < \alpha$ the set of β-inaccessible cardinals $< \kappa$ is unbounded.
We will say that a cardinal is hyperinaccessible if it is α-inaccessible for every ordinal α.

Proposition Every measurable cardinal is hyperinaccessible.

Proposition Vopěnka's principle implies the existence of arbitrarily large measurable cardinals.

Remark Vopěnka's principle is provably independent of ZFC + existence of measurable cardinals. In fact, it implies the existence of supercompact cardinals, which are a stronger version of strongly compact cardinals, but still much weaker than Vopěnka's principle.

- Our final point will be to establish conditions under which Vopěnka's principle can be satisfied.

Definition A model of a theory is transitive if it is transitive as a set, i.e. $x \in y \in M$ implies $x \in M$.

Definition Given two models of ZFC M_1 and M_2, a map $h : M_1 \to M_2$ is called an elementary embedding if for each formula $\phi(x_1, \ldots, x_n)$ and any assignment $x_i \mapsto a_i$ with $a_i \in M_1$, we have that $\phi(a_1, \ldots, a_n)$ is true in M_1 if and only if $\phi(ha_1, \ldots, ha_n)$ is true in M_2.

Definition A cardinal λ is called huge if there exists a transitive model M of ZFC and an elementary embedding $h : V \to M$ such that:

1. λ is the smallest ordinal such that $h(\lambda) \neq \lambda$;
2. M contains every function $h(\lambda) \to M$.

- This definition is quite technical. We won't be interested in all the details that come with it, rather we only want to focus on the following point:

Proposition Suppose λ is a huge cardinal, then W_λ is a model of ZFC in which Vopěnka's principle is true.

Remark This does not mean that the truthfulness of Vopěnka's principle follows from the existence of huge cardinals. It only means that if there is a huge cardinal, then we can find at least one model of ZFC in which the principle holds true, i.e. it is consistent with ZFC.