10. CVIČENIE

NORMA FUNKCIONÁLU

(1.) $C[0,1]$; f-cionál f:

$$f(u) := \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} u\left(\frac{1}{k}\right) \quad , \quad u \in C[0,1]$$

$\rightsquigarrow$ ukážte, že:

- f je lineárny a spojitý f-cionál
- norma f ?

R: $v, u \in C[0,1]$; $\alpha, \beta \in \mathbb{R}$

$$f(\alpha u + \beta v) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \left(\alpha u\left(\frac{1}{k}\right) + \beta v\left(\frac{1}{k}\right)\right)$$

$$= \alpha \cdot \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} u\left(\frac{1}{k}\right) + \beta \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} v\left(\frac{1}{k}\right)$$

$= \alpha f(u) + \beta f(v)$ ... f je lineárny f-cionál //

$\rightsquigarrow$ platí:

$$|f(u)| = \left| \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} u\left(\frac{1}{k}\right) \right| \leq \sum_{k=1}^{\infty} \left| \frac{(-1)^k}{k^2} u\left(\frac{1}{k}\right) \right| = \sum_{k=1}^{\infty} \frac{1}{k^2} \left|u\left(\frac{1}{k}\right)\right|$$

$$\left(\|u\| = \max_{x \in [0,1]} |u(x)| \right)$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{k^2} \|u\| = \|u\| \cdot \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \|u\| \quad \text{pre } \forall u \in C[0,1]$$

.... $|f(u)| \leq \frac{\pi^2}{6} \|u\|$

$\Rightarrow$ f je ohraničený f-cionál $\overset{\text{LINEARITA}}{\Rightarrow}$ f je spojitý f-cionál

• norma $\|f\|$?

$$\Rightarrow \quad \|f\| \le \frac{\pi^2}{6}$$

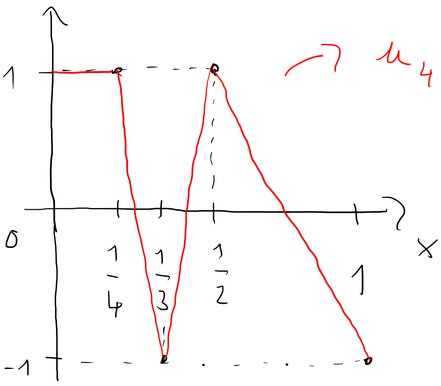
intuitívne by sme chceli zobrať f-ciu u takú, že:

$$u\left(\tfrac{1}{\ell}\right) = (-1)^\ell \quad \text{pre } \forall \ell \in \mathbb{N} \quad \dots \quad f(u) = \sum_{\ell=1}^{\infty} \frac{1}{\ell^2} = \frac{\pi^2}{6}$$

→ ale takéto $u \notin C[0,1]$

(u nie je spojitá v $x=0$)

⇝ môžeme ale uvažovať postupnosť $\{u_n\}_{n=1}^{\infty} \subseteq C[0,1]$:



(SAMY NÁJDITE PREDPIS PRE u_n PRE VŠEOBECNÉ n :))

• $\|u_n\| = 1$ pre $\forall n \in \mathbb{N}$; $u_n\left(\frac{1}{\ell}\right) = (-1)^\ell$; pre $\ell = 1, \dots, n$ | $u_n\left(\frac{1}{\ell}\right) = (-1)^n$ pre $\ell > n$

• $$f(u_n) = \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell^2} u_n\left(\tfrac{1}{\ell}\right) = \sum_{\ell=1}^{n} \frac{1}{\ell^2} + \sum_{\ell=n+1}^{\infty} \frac{(-1)^\ell}{\ell^2} (-1)^n$$

$$= \sum_{\ell=1}^{n} \frac{1}{\ell^2} + \left[\sum_{\ell=n+1}^{\infty} \frac{(-1)^\ell}{\ell^2}\right] \cdot (-1)^n \quad ; \forall n \in \mathbb{N}$$

• norma: $$\|f\| \ge |f(u_n)| = \left| \sum_{\ell=1}^{n} \frac{1}{\ell^2} + (-1)^n \cdot \left[\sum_{\ell=n+1}^{\infty} \frac{(-1)^\ell}{\ell^2} \right] \right|$$

pre $n \to \infty$ máme: $$\|f\| \ge \sum_{\ell=1}^{\infty} \frac{1}{\ell^2} = \frac{\pi^3}{6}$$ //

⇝ napokon platí: $\boxed{\|f\| = \frac{\pi^2}{6}}$

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2. PREDNÁŠKA — P8

$C[a,b]$; $f(u) = \int_a^b u(x)y(x)\,dx$, $u \in C[a,b]$

y je pevná f-cia z $C[a,b]$

⇝ platí: $\|f\| = \int_a^b |y(x)|\,dx$

⇝ DÔKAZ

- f je spojitý a lineárny ...

$$|f(u)| = \left| \int_a^b u(x)y(x)\,dx \right| \leq \int_a^b \underbrace{|u(x)|}_{\leq \|u\|} \cdot |y(x)|\,dx \leq \|u\| \cdot \int_a^b |y(x)|\,dx$$

$$\Rightarrow \quad \|f\| \leq \int_a^b |y(x)|\,dx$$

- dôkaz opačnej nerovnosti:

$\{u_n\} \subseteq C[a,b]$ .... $u_n(x) := \dfrac{y(x)}{|y(x)| + \frac{1}{n}}$; $n \in \mathbb{N}$

spojité f-cie, ktoré aproximujú $\operatorname{sgn} y(x)$

⇝ platí:

$$\|u_n\| = \max_{x \in [a,b]} |u_n(x)| = \max_{x \in [a,b]} \frac{|y(x)|}{|y(x)| + \frac{1}{n}} = \frac{\|y\|}{\|y\| + \frac{1}{n}} \quad ; \forall n \in \mathbb{N}$$

(pretože výraz $\frac{t}{t+\frac{1}{n}}$ je max. pre maximál. t ... je rastúca)

$\Rightarrow \quad \|u_n\| < 1$ pre $\forall n \in \mathbb{N}$

$\rightsquigarrow$ navyše platí : $\underline{\lim_{n\to\infty} \|u_n\| = 1}$

$\rightsquigarrow$ ukážeme, že platí : $\lim_{n\to\infty} |f(u_n)| = \int_a^b |g(x)|\,dx$

.... platí : $\left| |f(u_n)| - \int_a^b |g(x)|\,dx \right| < \frac{1}{n}(b-a)$

(POKÚSTE SA SAMY UKÁZAŤ :))

$\rightsquigarrow$ pre $n \to \infty$ dostávame danú limitu

$\rightsquigarrow$ platí : $\|f\| \geq |f(u_n)|$ pre $\forall n \in \mathbb{N}$

$$\Rightarrow \underline{\|f\|} \geq \lim_{n\to\infty} |f(u_n)| = \underline{\int_a^b |g(x)|\,dx}$$

$\rightsquigarrow$ teda naozaj platí : $\underline{\underline{\|f\| = \int_a^b |g(x)|\,dx}}$