PROBLEMS FOR THE COMPLEX ANALYSIS COURSE

*Problems marked by *** count for 6 points, those marked by ** count for 4 points, those marked by * count for 3 points, the others count for 2 points.*

1. Is there a holomorphic function $f(z)$ in $\{|z| < 1\}$ such that $f(1/n) = (-1)^n/n$ for $n \in \mathbb{N}$?

2. Is there a sequence of complex polynomials $P_n(z)$ which converges uniformly on the circle $\{|z| = 1\}$ to the function $f(z) = 1/z$?

3. Describe all harmonic functions u in a domain D such that u^3 is also harmonic.

4*. Let $f(z)$ be a holomorphic function in $\{0 < |z| < 1\}$ and the function $\sqrt{|z|}|f(z)|$ is bounded. Prove that f extends holomorphically to $\{|z| < 1\}$.

5.** Let $f(z)$ be a holomorphic function in $\{0 < |z| < +\infty\}$ and for all z we have

$$|f(z)| \leq \sqrt{|z|} + 1/\sqrt{|z|}.$$

Prove that f is constant.

6. Let $f(z) = u(z) + iv(z)$ be a holomorphic function in a domain D, and $u^2 + v^3 = 1$ holds. Prove that f is constant.

7*. Let $D = \{0 < |z| < 1\}$ and $f(z)$ be a holomorphic function in D which is continuous up to the boundary of D and vanishes at the boundary. Prove that f is indentically 0.

8. Describe all C^∞ functions $f(z)$ in $\mathbb{C}$, such that $\frac{\partial f}{\partial \bar{z}} = z$.

9. For which values of $R > 0$ is the set $\{|z^2 - 1| < R\}$ connected?

10. Find the residue at 0 for the function $f(z) = \exp(\operatorname{ctg} z)$.

11*. Let $f(z)$ be a holomorphic function in $\mathbb{C}$ and $F(\mathbb{R}) \subset \mathbb{R}$. Prove that $f(\bar{z}) = \overline{f(z)}$ for all z.

12.** By considering integrals over circles $\{|z| = \pi n + \pi/2\}$ from the function $f(\xi) = \frac{\operatorname{ctg} \xi}{\xi(\xi - z)}$, prove the expansion

$$\operatorname{ctg} z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z - \pi n} + \frac{1}{z + \pi n} \right)$$

for all $z \neq \pi n$, $n \in \mathbb{Z}$.

13. Can the function $f(x) = x \ln(1 + x)$ be extended holomorphically from the positive ray $\mathbb{R}^+$ to a domain in complex plane? To the entire complex plane?

14*. Does there exist a function $f(z)$ holomorphic in $\{|z| > 0\}$ such that for all z we have $|f(z)| > \exp(1/|z|)$?

15*. Prove that the function $f(z) := \sum_{n \geq 1} z^{n!}$ is holomorphic in the disc $\{|z| < 1\}$ but cannot be extended holomorphically to a neighborhood of any point a in the boundary of the disc.

16*. Let $y(x)$ be the solution of the differential equation $y' = y^2 + e^{-1/x^2}$ with the initial value $y(0) = 0$. Can $y(x)$ be extended holomorphically from the real line to the complex plane? (The function e^{-1/x^2} here is extended smoothly to 0 with the valued 0).

17*. Prove that if a sequence of holomorphic polynomials converges uniformly in the boundary of the disc $D = \{|z| < 1\}$, then it converges uniformly in the whole closed disc.

18.** Let $f(z)$ be a continuous function in $\mathbb{C}$, which is holomorphic in $\mathbb{C} \setminus \mathbb{R}$. Prove that f is holomorphic in $\mathbb{C}$.

19. Prove the Minimum Modulus Principle: if $f(z)$ is a *nonvanishing* function holomorphic in a domain D such that $|f(z)|$ has a local minimum at a point $a \in D$, then f is constant.

20*. Let $u(z)$ be a harmonic function $f(z)$ in the annulus $\{1 < |z| < 2\}$ which equals 0 on the internal and 1 on the external circles. Prove that $u(z) = a \ln |z|$ for an appropriate a.

21*.** Let $f \in \mathcal{O}(B_1(0))$, $f(0) = 0$, $f'(0) = 1$. Denote the image domain of $B_1(0)$ under f by Ω. Prove that $\lambda(\Omega) \geq \pi$, where λ denotes the area (so, the area of the initial domain can only increase!).

Hint. Use the Mean Value Theorem.

22. Prove that there exists a holomorphic function $f(z)$ in $\{|z| > 1\}$ such that for every z, $f(z)$ equals to one of the values of $\sqrt{1+z^2}$.

23*. Prove that if $f(z)$ is a holomorphic function in $\{|z| < 1\}$ and $\operatorname{Im} f(z) > 0$ for all z, then there exists a holomorphic function $g(z)$ in the same domain such that $f(z) = e^{g(z)}$.

24*. Let $u(x, y)$ be a harmonic polynomial. Prove that it is the real part of a complex polynomial.

25*. Can the function $f(z) = 1/z^2$ be approximated by a normally converging sequence of complex polynomials in the domain $\{1 < |z| < 2\}$?

Hint. Argue as in Problem 2.

26. A function $f(z)$ is holomorphic in $\{|z| > 1\}$ and is bounded there from below: $|f(z)| \geq M > 0$ for all z. Prove that there exists a (finite or infinite) $\lim_{z\to\infty} f(z)$.

27. Construct a biholomorphic map from $B_1(0)$ onto the domain $\{\operatorname{Im} z > \operatorname{Re} z\}$.

28*. Prove that the group of linear-fractional automorphisms of the upper half-plane $\{\operatorname{Im} z > 0\}$ consists of the maps

$$z \mapsto \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R},\ ad - bc \neq 0.$$

29*. Let u be a harmonic function in the right half-plane $\{\operatorname{Re} z > 0\}$, continuous up to the boundary, which is zero on the boundary and has the zero limit at ∞. Prove that $u \equiv 0$.

Hint. Use linear-fractional transformations.

30*. Let u be a bounded harmonic function in the annulus $D = \{0 < |z| < 1\}$. Assume, in addition, that the harmonically conjugated function v is single-valued in D. Prove that u extends harmonically to the unit disc $B_1(0)$.

Hint. Argue as in the proof of the maximum principle.