SOLUTIONS FOR ASSIGNMENT 4 ($n = 1$)

1. Problem 1

For each of the following functions, determine all its zeroes and all its isolated singular points *in the extended complex plane* $\overline{\mathbb{C}}$ (that is, the point ∞ must be investigated as well!). As long as a point is a zero or a pole, determine the order. Explain your answer!

a) $f(z) = \frac{1-\cos(z^6)}{(\sin z)^5} \cdot e^{\frac{1}{\pi - z}}$

b) $f(z) = \sqrt{z} \sin\left(\frac{1}{\sqrt{z^7}}\right)$ (explain also why the function is holomorphic outside its isolated singular points; the choice of $\operatorname{Arg} z$ for the two roots is the same).

Solution for a). First of all, the ∞ is *not* an isolated singularity of $f(z)$ (and not an isolated zero), because zeroes of the denominator $z = \pi k$ accumulate to ∞. So, this is the verdict for $z = \infty$: *not an isolated singularity and not an isolated zero.*

Potential zeroes are the points where $\cos(z^6) = 1$, that is $z^6 = 2\pi k$ and $z = z_{k,l} = \sqrt[6]{2\pi k} e^{\pi i l/3}$, $k > 0$, $l = 0, ..., 5$ or $z = z_{k,l} = \sqrt[6]{-2\pi k} e^{i(2\pi l + \pi)/6}$, $k < 0$, $l = 0, ..., 5$ (you can use alternative representations, of course). The value $k = 0$ shall be considered separately. For nonzero k, we see that the exponent and the *sin* are holomorphic near $z_{k,l}$, and the *sin* does not vanish, so $z_{k,l}$ are true zeroes of $f(z)$. Let us determine their order. The easiest way is as follows. Since the exponent and $1/(\sin z)^5$ are holomorphic and do not vanish at $z_{k,l}$ (IMPORTANT!), they do *not* effect the order. Hence, we just look at $1 - \cos(z^6)$. We easily find that the first derivative $6z^5 \sin(z^6)$ of the latter function vanishes at $z_{k,l}$, while the second derivative $30z^4 \sin(z^6) + 36z^{10} \cos(z^6)$ does not.

Hence, *$z_{k,l}$ are all zeroes of $f(z)$ of order* 2.

Now let us deal with $z = 0$. The exponent is holomorphic and does not vanish at 0 (IMPORTANT!), so it does *not* effect the order. Now we use the classical Taylor expansions:

$$1 - \cos(z^6) = z^{12}/2 + ... = z^{12}(0.5 + ...), \quad \sin z = z + ... = z(1 + ...),$$

so

$$\frac{1 - \cos(z^6)}{(\sin z)^5} = z^7(0.5 + ...)/(1 + ...) = 0.5z^7 + ...$$

(dots all the time denote a holomorphic near 0 sum of terms of higher order), so we conclude that $z = 0$ *is a removable singularity which is at the same tine a zero of $f(z)$ of order* 7.

Now we deal with the other isolated singularities. They corresponds to $\sin z = 0$, $z \neq 0$, so $z = z_k = \pi k$, $k \neq 0$. The case $k = 1$ will be considered separately, while for $k \neq 1$ the factors $1 - \cos(z^6)$ and the exponent are holomorphic and do not vanish at z_k, so they do not effect the nature of the singularity, and we just deal with $g(z) = \frac{1}{(\sin z)^5}$. The easiest way now is to use the property: if $h(z)$ has at a point a a zero of order l, then $1/f(z)$ has at a a pole of order l, and also the fact that the order of product is the sum of orders. From $\sin z = z(1 + ...)$ we know that $\sin z$ has a zero of order 1 at $z = 0$, but $\sin z$ is (almost) periodic with period π: $\sin(z + \pi k) = (-1)^k \sin z$ (!), so it also has a zero of order 1 at each z_k. Alternatively, one can again use derivatives.

So, we finally see that *all* z_k *with* $k \neq 1$ *are poles of* $f(z)$ *of order* 5.

It remains to deal with $z = \pi$. We remember that from the Taylor series of exponent,

$$e^{1/(\pi-z)} = \sum_{j=0}^{\infty} \frac{1}{(\pi - z)^j j!}.$$

This series (which becomes the Laurent series for $e^{1/(\pi-z)}$ at $z = \pi$) has infinitely many negative terms, and this implies that $e^{1/(\pi-z)}$ has at $z = \pi$ an essential singularity. Now we shall still prove (!) that $f(z)$ has the same kind of singularity (counterexample: $e^{z+1/z} \cdot e^{-1/z}$ has a removable singularity at 0, while both factors have an essential singularity!). To see this, we express the exponent:

$$e^{1/(\pi-z)} = f(z)(\sin z)^5/(1 - \cos(z^6)).$$

The factor $(\sin z)^5$ is holomorphic near 0 and the factor $1/(1 - \cos(z^6))$ has a pole at 0, so if $f(z)$ had a removable singularity or a pole at 0, then the product $e^{1/(\pi-z)} = f(z)(\sin z)^5/(1 - \cos(z^6))$ would also have a removable singularity or a pole at 0, which is a contradiction.

So, $z = \pi$ *is an esential singularity.*

Solution for a). First of all, from the classical Taylor expansion of $\sin z$ we get:

$$f(z) = \sqrt{z} \left(\sum_{j=0}^{\infty} \left(\frac{1}{\sqrt{z^7}} \right)^{2j+1} \frac{(-1)^j}{(2j+1)!} \right) = \sum_{j=0}^{\infty} \left(\frac{1}{z} \right)^{7j+3} \frac{(-1)^j}{(2j+1)!},$$

which proves that $f(z)$ *is single-valued and holomorphic in* $\mathbb{C} \setminus \{0\}$.

Let us start with $z = \infty$: setting $w = 1/z$, $f(z) = g(w)$, we see that

$$g(w) = \sum_{j=0}^{\infty} w^{7j+3} \frac{(-1)^j}{(2j+1)!},$$

so that $g(w)$ is simply *holomorphic* at $w = 0$, hence $z = \infty$ *is a removable singularity which is at the same time a zero of order* 3.

For the isolated singularity $z = 0$, we see from the above that the Laurent series of $f(z)$ at $z = 0$ has infinitely many negative terms, hence $z = 0$ *is an essential singularity.*

It remains to deal with the isolated zeroes. They correspond to $\sin\left(\frac{1}{\sqrt{z^7}}\right) = 0$, so we get $z = z_{k,l} = \frac{1}{\sqrt[7]{\pi^2 k^2}} e^{2\pi i l/7}$, $k > 0$, $l = 0, ..., 6$, or $z = z_{k,l} = \frac{1}{\sqrt[7]{\pi^2 k^2}} e^{i(\pi+2\pi l)/7}$, $k < 0$, $l = 0, ..., 6$. To determine the order at $z_{k,l}$, we first note that the factor $\sqrt{z}$ is holomorphic and nonvanishing *near each fixed* $z_{k,l}$. So, it does not effect the order. Now, for the remaining function $\sin\left(\frac{1}{\sqrt{z^7}}\right)$, we compute the derivative $z^{-9/2} \cos\left(\frac{1}{\sqrt{z^7}}\right)$ and see that it is nonzero at $z_{k,l}$. Hence, *each* $z_{k,l}$ *is a zero of* $f(z)$ *of order* 1.

2. Problem 2

Find Taylor/Laurent expensions of the function $f(z)$ in *all* (!) possible discs and annuli with the center $z = 0$:

$$f(z) = \frac{z^4}{z(z-1)(z-2)}$$

After doing so, determine the types of isolated singularity at the points $z = 0$ and $z = \infty$ (including the order, if applicable).

Solution. The function has the following singularities in $\mathbb{C}$: $0, 1, 2$. In view of that, there are three principal annuli where it admits a Laurent expansion:

$$D_1 = \{0 < |z| < 1\},\ D_2 = \{1 < |z| < 2\},\ D_3 = \{2 < |z| < +\infty\},$$

and all the other annuli are contained in one of the D_j. So, it is enough to provide the Laurent expansion in each of the D_j.

We will make use of the partial fraction decomposition:

$$f(z) = z^3 \left(\frac{1}{(z-1)(z-2)} \right) = z^3 \left(\frac{1}{z-2} - \frac{1}{z-1} \right).$$

Let us start with D_1. Since in D_1 we have $|z| < 1$, we can use the geometric progression and get:

$$\frac{1}{z-1} = -\sum_{j=0}^{\infty} z^j, \quad \frac{1}{z-2} = -\frac{1/2}{1-z/2} = -\sum_{j=0}^{\infty} \frac{1}{2^{j+1}} z^j.$$

Now

$$f(z) = z^3 \left(\sum_{j=0}^{\infty} z^j - \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} z^j \right) = \sum_{j=0}^{\infty} \left(1 - \frac{1}{2^{j+1}} \right) z^{j+3}.$$

We also see from here that $z = 0$ *is a removable singularity for* $f(z)$*, which is at the same time an isolated zero of order* 3.

We deal next with D_2. Here $|z/2| < 1$, $|1/z| < 1$, so still using simply a geometric progression we get:

$$\frac{1}{z-1} = \frac{1/z}{1-1/z} = \sum_{j=0}^{\infty} \frac{1}{z^{j+1}}, \quad \frac{1}{z-2} = -\frac{1/2}{1-z/2} = -\sum_{j=0}^{\infty} \frac{1}{2^{j+1}} z^j,$$

$$f(z) = z^3 \left(-\sum_{j=0}^{\infty} \frac{1}{z^{j+1}} - \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} z^j \right) = -\sum_{j=0}^{\infty} \frac{1}{z^{j-2}} - \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} z^{j+3}.$$

It is a good example when a Laurent series actually goes from $-\infty$ to $+\infty$.

Finally, we deal with D_3. Here $|2/z| < 1$, $|1/z| < 1$, so:

$$\frac{1}{z-1} = \frac{1/z}{1-1/z} = \sum_{j=0}^{\infty} \frac{1}{z^{j+1}}, \quad \frac{1}{z-2} = \frac{1/z}{1-2/z} = \sum_{j=0}^{\infty} \frac{2^j}{z^{j+1}},$$

$$f(z) = \sum_{j=0}^{\infty} \frac{1}{z^{j-2}} \left(2^j - 1\right) = \sum_{j=1}^{\infty} \frac{1}{z^{j-2}} \left(2^j - 1\right).$$

Also, by setting $w = 1/z$, $f(z) = g(w)$, we see that the Laurent expansion of $g(w)$ at $w = 0$ has exactly one negative term which is $\frac{1}{w}$, and from here $z = \infty$ *is a pole of order* 1 *for* $f(z)$.

3. Problem 3

Everybody succeeded, so I will not provide a solution here.