Prednaska 21

Monday, May 10, 2021 2:12 PM

3) $f(z) = \dfrac{\operatorname{ctg} z}{z^2}$, $\operatorname{res}_0 f(z) = ?$

$$\operatorname{ctg} z = \frac{\cos z}{\sin z} = \frac{1 - \frac{z^2}{2!} + \dots}{z\left(1 - \frac{z^2}{3!} + \dots\right)} = \frac{h(z)}{z}, \quad h(0) \neq 0$$

$$f(z) = \frac{h(z)}{z^3} \Rightarrow z = 0: \text{ pol rádu} = 3$$

<u>Metoda 1</u>: $\operatorname{res}_0 f = \dfrac{1}{2!} \lim\limits_{z \to 0} \dfrac{d^2}{dz^2}\left(z \operatorname{ctg} z\right) = \dots$

<u>Metoda 2</u>: $\operatorname{ctg} z = \dfrac{\cos z}{\sin z} = \dfrac{1 - \frac{z^2}{2} + O(z^3)}{z - \frac{z^3}{6} + O(z^4)} =$

$$= \frac{1}{z}\left(1 - \frac{z^2}{2} + O(z^3)\right) \frac{1}{1 - \underbrace{\frac{z^2}{6} + O(z^3)}_{t}} = \left| \text{geom postup } \frac{1}{1-t} \right|$$

$$= \frac{1}{z}\left(1 - \frac{z^2}{2}\right)\left(1 + \left(\frac{z^2}{6} + O(z^3)\right) + O(z^4)\right) =$$

$$= \frac{1}{z} + z\left(\frac{1}{6} - \frac{1}{2}\right) + O(z^2)$$

$$\Rightarrow f(z) = \frac{1}{z^3} - \frac{1}{3}\frac{1}{z} + \text{hol} \Rightarrow \operatorname{res}_0 f = -\frac{1}{3}$$

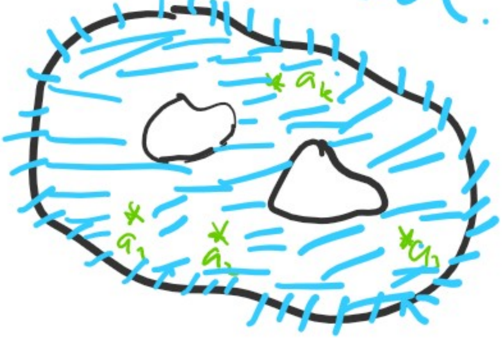
4) $f = \dfrac{\operatorname{ctg} z}{z^3}$, $\operatorname{res}_0 f = ?$

$f(-z) = -f(z)$ ...

$f(-z) = f(z)$ - symetr. (sudá funkce)
$\Rightarrow c_{2k-1} = 0 \ \forall k \Rightarrow c_{-1} = \operatorname{res}_0 f = 0.$

Reziduová Věta: Nechť $\mathcal{D} \subset \mathbb{C}$ je n-souvislá napravitelná Jordanova oblast $(n \geq 1)$;
$\Omega = \mathcal{D} \setminus \{a_1, .., a_k\}$; nechť $f \in \mathcal{O}(\Omega)$, která taky je hol v okolí $\partial\mathcal{D}$ (zejména, v bod a_j - isol. sing. bod pro f).

$f \in$ hol.



Tak to:

$$\boxed{\int_{\partial\mathcal{D}} f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname{res}_{a_j} f(z)}$$

Důkaz: Vybereme $\overline{B_{\varepsilon_j}(a_j)} \subset \mathcal{D}$, který neprotíná se (každý 2)

G

Uvažujeme $G := \mathcal{D} \setminus \overline{\bigcup_{j=1}^{k} B_{r_j}(a_j)}$ –

– $(n+k)$-souvislá Jord. oblast, napravitelná; $G \subset \Omega \Rightarrow f \in \mathcal{O}(G)$

$\partial G = \partial \mathcal{D} \cup C_{r_j}(a_j)$

navíc

$f \in \mathcal{O}(\overline{G}) \Rightarrow$

Můžeme přimenit Větu Cauchy pro f v G:

$$\int_{\partial G} f(z)\,dz = 0$$

$$\int_{\partial \mathcal{D}} f(z)\,dz - \sum_{j=1}^{k} \int_{C_{r_j}(a_j)} f(z)\,dz = 0$$

$$\Rightarrow \int_{\partial \mathcal{D}} f(z)\,dz = \sum_{j=1}^{k} \int_{|z-a_j|=r_j} f(z)\,dz =$$

$$= \sum_{j=1}^{k} 2\pi i \cdot \operatorname{res}_{a_j} f(z) = 2\pi i \sum_{j=1}^{k} \operatorname{res}_{a_j} f(z).$$

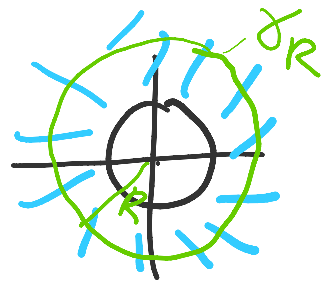
$\blacksquare$

Má hodně aplikací

v analýze, teorie čísel, atd!

Lemma 1: nechť $f \in C(\{|z|>R_0\})$

Nechť navíc $|f(z)| = O\left(\frac{1}{|z|^2}\right)_{z\to\infty}$



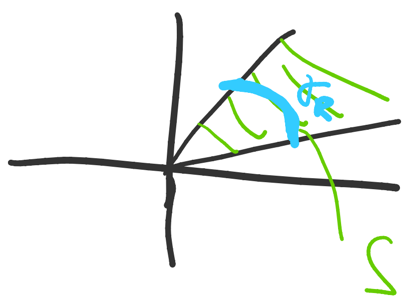
$$\Rightarrow \int_{\gamma_R} f(z)\,dz \xrightarrow{R\to\infty} 0$$

Důkaz: $\left|\int_{\gamma_R} f(z)\,dz\right| \leq 2\pi R \cdot \max_{|z|=R} |f(z)| =$

$$= 2\pi R \cdot O\left(\frac{1}{R^2}\right) \leq \frac{M_0}{R}$$

$$\Rightarrow \int_{\gamma_R} f(z)\,dz \xrightarrow{R\to\infty} 0$$

Důsledek: stejný fakt platí v případě $f \in C(S)$, $|f| = O\left(\frac{1}{|z|^2}\right)$, důkaz je stejný



Poznámka: Stejné platí když $\lim_{z\to\infty} z f(z) = 0$

Poznámka: stejně platí když $\lim_{z\to\infty} z f(z) = 0$

(stejný důkaz)

Integrály od racion. funkce

Příklad: $\int\limits_{-\infty}^{+\infty} \frac{dx}{x^4+1}$; uvažujeme $f(z)=\frac{1}{z^4+1}$

$f \in C(\mathbb{R})$; ale v $\mathbb{C}$ má sing body:

$z^4 = -1 = e^{\pi i + 2\pi k i} \Rightarrow z = \left\{ e^{\frac{\pi i + 2\pi k i}{4}} \right\}_{k=0,1,2,3}$

$z_0 = e^{\frac{\pi i}{4}}$, $z_1 = e^{\frac{3\pi i}{4}}$, $z_2 = e^{\frac{5\pi i}{4}}$, $z_3 = e^{\frac{7\pi i}{4}}$

Uvažujeme oblast D_R: – půlka kruhu

($R \to \infty$)

f má v D 2 sing body z_0, z_1, jinak je hol ve všech bodech $\overline{D_R}$ $\Rightarrow$ můžeme přímenit k f a D_R Rez.

Věta: $\int\limits_{\partial D_R} f(z)\,dz = 2\pi i \left(\operatorname{res}_{z_0} f(z) + \operatorname{res}_{z_1} f(z) \right)$.

$\forall R$!

$\ldots\, \mathrm{res}_{z_0} \ldots z_1)$.

∂D_R $\qquad \forall R!$

$$\int_{\substack{|z|=R \\ \mathrm{Im}\, z \geq 0}} F(z)\,dz + \int_{-R}^{R} F(z)\,dz$$

lim = 0 podle lemmi 1!

$= \lim = \int_{-\infty}^{\infty} f(x)\,dx$

$f(z) = \frac{1}{z^4+1} = O\left(\frac{1}{z^4}\right)_{z\to\infty}$

Ted přimenime $\lim\limits_{R\to\infty}$;

$$\boxed{0 + \int_{-\infty}^{+\infty} \frac{dx}{x^4+1} = 2\pi i \left(\mathrm{res}_{z_0} f + \mathrm{res}_{z_1} f \right)}$$

v okolí z_0: $f(z) = \frac{1}{(z-z_0)(z-z_1)(z-z_2)(z-z_3)} \Rightarrow$

$z = z_0$ – pól řádu $k = 1$

$$\Rightarrow \mathrm{res}_{z_0} f = \lim_{z\to z_0} (z-z_0) f(z) =$$

$$= \frac{1}{(z_0-z_1)(z_0-z_2)(z_0-z_3)} = |\text{trigon. tvar}| =$$

$$= \frac{1}{2\cos\frac{\pi}{4}} \, \frac{1}{2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)} \, \frac{1}{2i\sin\frac{\pi}{4}} = \frac{1}{2i} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{1+i}$$

analog $\mathrm{res}_{z_1} f = \frac{1}{(z_1-z_0)(z_1-z_2)(z_1-z_3)} =$

1

$$= \frac{1}{-2\cos\frac{\pi}{4}\cdot 2i\sin\frac{3\pi}{4}\cdot 2\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)} =$$

$$= \frac{1}{-2i}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{-1+i} \Rightarrow$$

$$\int_{-\infty}^{+\infty}\frac{dx}{1+x^4} = 2\pi i\left(\frac{1}{2i}\frac{1}{\sqrt{2}}\frac{1}{1+i} - \frac{1}{2i}\frac{1}{\sqrt{2}}\frac{1}{i-1}\right) =$$

$$= \frac{\pi}{\sqrt{2}}\left(\frac{1}{1+i}+\frac{1}{1-i}\right) = \frac{\pi}{\sqrt{2}}\cdot\frac{2}{1-i^2} = \frac{\pi}{\sqrt{2}}$$

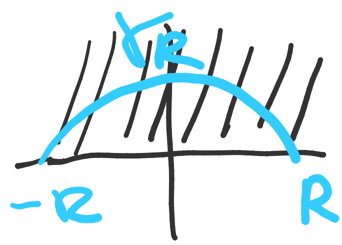
Pro ostatní rac. funkce

$R = \frac{P}{Q}$, $\deg Q \geq 2 + \deg P$

fungují stejně

<u>Lemma 2</u> (Jordanova Lemma)

Nechť $f \in C(\{\operatorname{Im} z \geq 0\})$,



a navíc $f(z) \xrightarrow[\operatorname{Im} z \geq 0]{z\to\infty} 0$

(např. platí když $|f| = O\left(\frac{1}{|z|}\right)$)

Pak, $\int f(z) e^{i\lambda z}\, dz \xrightarrow{R\to\infty} 0$ $(\lambda > 0)$

Pak, $\int_{\gamma_R} f(z)e^{i\lambda z}\,dz \xrightarrow{R\to\infty} 0$ $\left(\lambda>0\right)$ $\forall\lambda$

<u>Důkaz</u>: $\int_{\gamma_R} f(z)e^{i\lambda z}\,dz = \left| z = Re^{it},\ 0\leq t\leq\pi \right| =$

$$= \int_0^\pi f(Re^{it})\, e^{i\lambda Re^{it}} \cdot Rie^{it}\,dt$$

$i\lambda R(\cos t + i\sin t)$

$|e^{x+iy}| = e^x$

$$\Rightarrow \left|\int_{\gamma_R} f(z)e^{i\lambda z}\,dz\right| = R\left|\int_0^\pi f(Re^{it})e^{i\lambda Re^{it}}\cdot e^{it}\,dt\right| \leq$$

$$\leq R\int_0^\pi |f(Re^{it})|\cdot e^{-\lambda R\sin t}\,dt \leq$$

$$\leq R\cdot \underset{\substack{\max|f| \\ |z|=R \\ \operatorname{Im} z\geq 0}}{M(R)}\cdot 2\int_0^{\pi/2} e^{-\lambda R\sin t} \leq \left\{ \begin{array}{l} \sin t \geq \frac{2}{\pi}t \\ 0\leq t\leq \frac{\pi}{2} \Rightarrow \\ -\sin t \leq -\frac{2}{\pi}t \end{array} \right\}$$

$$\leq 2RM(R)\int_0^{\pi/2} e^{-\lambda Rt}\,dt =$$

$$= 2\cancel{R}M(R)\cdot\frac{-1}{\lambda \cancel{R}}\, e^{-\lambda Rt}\Big|_0^{\frac{\pi}{2}} =$$

$$= \frac{2}{\lambda}\left(1-e^{-\lambda R\frac{\pi}{2}}\right)\cdot M(R) \xrightarrow{R\to\infty} 0$$

protože $M(R)\to 0$ $\blacksquare$

Příklad: $\int^{+\infty} \frac{\cos x}{\ldots}\,dx = \{\cos x = \operatorname{Re} e^{ix}\}$

Příklad: $\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+1}\,dx = \{\cos x = \mathrm{Re}\, e^{ix}\} =$

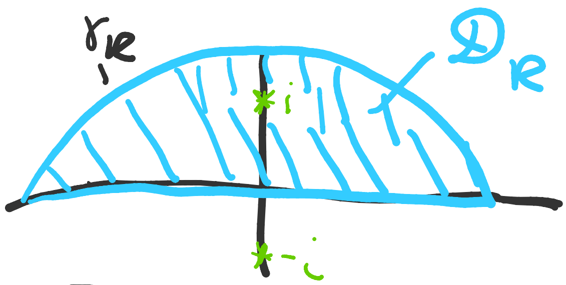
$$= \mathrm{Re} \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+1}\,dx \; ; \quad F(z) = \frac{e^{iz}}{z^2+1}$$

$$|f(x)| = \frac{1}{x^2+1}$$

$\int_{-\infty}^{+\infty} F(z)\,dz$, $\quad f(z)$ je hol v bodech v $\mathbb{C}$,

Zejména: $z^2 = -1 \Leftrightarrow z_1 = i,\ z_2 = -i$

Můžeme přímat Rez.



Věta:

$$\int_{\partial D_R} F(z)\,dz = 2\pi i \cdot \mathrm{res}_i\, F(z)$$

$$\| \quad \int_{\gamma_R} F(z)\,dz + \int_{-R}^{R} F(z)\,dz$$

Pak, bereme $\lim_{R\to\infty}$:

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = \lim_{R\to\infty} \int_{\gamma_R} \frac{1}{z^2+1} \cdot e^{iz}\,dz = 0$$

podle Lemmi Jordana $\left(\frac{1}{z^2+1} \xrightarrow{z\to\infty} 0\right)$.

podle Lemmi rovnou $\left(\frac{1}{z^2+1} \to 0\right)$.

$$\lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz = \int_{-\infty}^{+\infty} f(z)\,dz \implies$$

$$\int_{-\infty}^{+\infty} \frac{\cos x\,dx}{x^2+1} = \operatorname{Re}\left(2\pi i \operatorname{res}_i f(z)\right)$$

$$f(z) = \frac{e^{iz}}{z^2+1} = \frac{e^{iz}}{(z-i)(z+i)} \implies z=i \text{ je pol řádu} = 1$$

$$\operatorname{res}_i f(z) = \lim_{z\to i} \frac{e^{iz}}{z+i} = \frac{e^{-1}}{2i}$$

$$\implies \int_{-\infty}^{+\infty} \frac{\cos x}{x^2+1}\,dx = \operatorname{Re}\left(2\pi i \cdot \frac{e^{-1}}{2i}\right) = \frac{\pi}{e}$$

Lemma 3: Nechť a je pol řádu $k=1$ pro hol. funkce f. Pak platí:

$$\lim_{\varepsilon\to 0} \int_{\underbrace{\left\{\substack{|z-a|=\varepsilon \\ \operatorname{Im}(z-a)\geq 0}\right\}}_{=\gamma_\varepsilon}} f(z)\,dz = \pi i \cdot \operatorname{res}_a f(z)$$

Důkaz: kvadratujeme Laur. rozvoj

$$f(z) = \frac{c_{-1}}{z-a} + \sum_{n=0}^{\infty} c_n (z-a)^n$$

konverguje stejnom. na kompaktech $\implies$

[illegible] $\int$:

přimeníme $\int_{\gamma_\varepsilon}$:

$$\int_{\gamma_\varepsilon} f(z)\,dz = c_{-1}\int_{\gamma_\varepsilon}\frac{dz}{z-a} + \sum_{n=0}^{\infty} c_n \int_{\gamma_\varepsilon}(z-a)^n dz$$

(1) (2)

(1) $\int_{\gamma_\varepsilon}\frac{dz}{z-a} = \left| z = a+\varepsilon e^{it},\ 0<t\le\pi \right| = \int_0^\pi \frac{1}{\varepsilon e^{it}}\cdot\varepsilon i e^{it}\,dt = \pi i$

(2) $\int_{\gamma_\varepsilon}(z-a)^n dz = \left| \quad \right| = \int_0^\pi \varepsilon^n e^{itn}\cdot i\varepsilon e^{it} =$

$$= \frac{i\varepsilon^{n+1}}{i(n+1)} e^{itn}\Big|_0^\pi = \frac{\varepsilon^{n+1}}{n+1}\left((-1)^n - 1\right)$$

$$\Rightarrow \lim_{\varepsilon\to 0}\sum_{n=0}^{\infty} c_n \int_{\gamma_\varepsilon}(z-a)^n dz = 0$$

protože řada z mod:

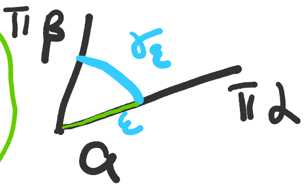
$$\varepsilon\cdot\sum |c_n|\varepsilon^n \text{ (o.k. } 2) \xrightarrow{\varepsilon\to 0} 0$$

bereme $\lim_{\varepsilon\to 0}$: $\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon} f(z) = \pi i\cdot \underset{\text{“}\operatorname{res}_a f(z)\text{”}}{c_{-1}}$

Poznámka: stejný výsl. platí pro $\forall$ jiné polokružnice

∀ jinn potokuzněnic

Poznamka: podobne, když a je pol řádu $=1$ pro f, $\gamma_\varepsilon = \left\{ \begin{array}{l} |z-a| = \varepsilon \\ \pi\alpha < \arg(z-a) < \pi\beta \end{array} \right\}$



$$\Rightarrow \boxed{\lim_{\varepsilon\to 0} \int_{\gamma_\varepsilon} f(z)\,dz = \pi i(\beta-\alpha)\cdot \operatorname{res}_a f(z)}$$

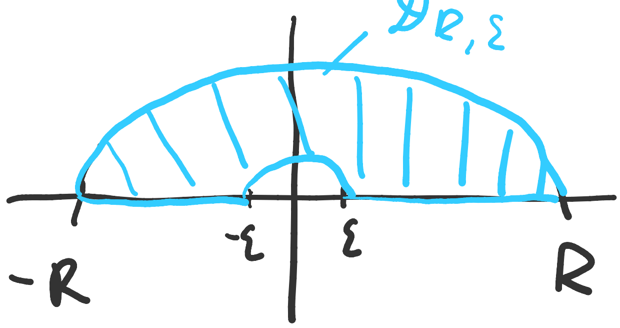
(lemma 3: $\alpha = 0$, $\beta = 1$)

$\frac{\beta-\alpha}{2} \operatorname{res}_a f$ – lomený reziduum

lemma o lomeném reziduu.

Příklad: Integral Dirichlet

$$I = \int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx = ?$$



$I = \lim I_{R,\varepsilon}$,

$$I_{R,\varepsilon} = \int_{-R}^{-\varepsilon} \frac{\sin x}{x}\,dx + \int_{\varepsilon}^{R} \frac{\sin x}{x}\,dx =$$

$e^{ix} = \cos x + i\sin x$
$\sin x = \operatorname{Im}(e^{ix})$

$$= \operatorname{Im}\left(\int^{-\varepsilon} \frac{e^{iz}}{z}\,dz + \int^{R} \frac{e^{iz}}{z}\,dz \right)$$

$$= \operatorname{Im}\left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \frac{e^{iz}}{z}\,dz \right)_{z\in\mathbb{R}}$$

Uvažujeme $\displaystyle\int_{\partial D_{R,\varepsilon}} \frac{e^{iz}}{z}\,dz = 0$ (nejsou sing. body v $D_{R,\varepsilon}$)

$$= \int_{\substack{|z|=R\\ \operatorname{Im} z \geq 0}} \frac{e^{iz}}{z}\,dz - \int_{\substack{|z|=\varepsilon\\ \operatorname{Im} z \geq 0}} \frac{e^{iz}}{z}\,dz + \int_{-R}^{-\varepsilon} \frac{e^{iz}}{z}\,dz + \int_{\varepsilon}^{R} \frac{e^{iz}}{z}\,dz$$

$R \to \infty$: 0 (podle lemmi 2)

$\pi i \operatorname{res}_0 \frac{e^{iz}}{z}$ (podle lemmi 3) $z=0$ - pól řádu $=1$

co nás zajímá

$$\operatorname{res}_0 \frac{e^{iz}}{z} = \lim_{z\to 0} e^{iz} = 1 \Rightarrow \text{v rovnice}$$

$\displaystyle\int_{\partial D_{R,\varepsilon}} \frac{e^{iz}}{z}\,dz = 0$ bereme $\displaystyle\lim_{R\to\infty,\ \varepsilon\to 0}$:

$$0 = 0 - \pi i \cdot 1 + \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{-R}^{-\varepsilon} \frac{e^{iz}}{z}\,dz + \int_{\varepsilon}^{R} \frac{e^{iz}}{z}\,dz$$

$\Rightarrow$ bereme Im: $\lim I_{R,\varepsilon} = I = \pi$

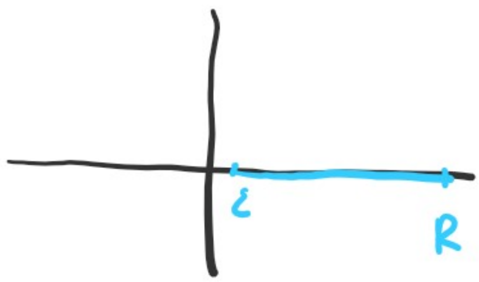
$= /$ vezmeme Im. $\lim I_{R,\varepsilon} = I = \pi$

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x}\, dx = \pi$$

Integrály od mnohoznačních funkce

Příklad: $\displaystyle\int_0^{+\infty} \frac{\sqrt{x}}{(1+x)^2}\, dx = ?$

$$= \lim_{\substack{R \to +\infty \\ \varepsilon \to 0}} \underbrace{\int_\varepsilon^R \frac{\sqrt{x}\, dx}{(1+x)^2}}_{= I_{R,\varepsilon}}$$

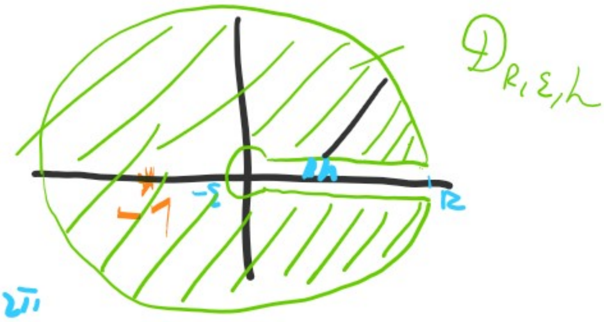


Uvažujeme:

$$\int_{\partial D_{R,\varepsilon,h}} \frac{\sqrt{z}}{(1+z)^2}\, dz$$

řez $\mathbb{R}^+$

$0 < \operatorname{Arg} z < 2\pi$



$f(z) \in O(D_{R,\varepsilon,h} \setminus \{z = -1\})$

isol. sing. bod

$$\Rightarrow \int_{\partial D_{R,\varepsilon,h}} f(z) = 2\pi i \cdot \operatorname{res}_{-1} f(z) = |z = -1 \text{ - pól řádu } = 2| =$$

$$= \frac{1}{1!} 2\pi i \lim_{z \to -1} (\sqrt{z})' = 2\pi i \cdot \frac{1}{2\sqrt{z}}\Big|_{z=-1} = \pi i \cdot \frac{1}{e^{\pi i/2}} = \pi$$

$$= \frac{1}{1!} 2\pi i \lim_{z\to -1} (\sqrt{z})' = 2\pi i \cdot \frac{1}{2\sqrt{z}}\Big|_{z=-1} = \pi i \cdot \frac{1}{e^{\pi i/2}} = \pi$$

$$-1 = e^{\pi i}$$

$$\Rightarrow \lim_{h\to 0} \int_{\partial D_{R,\varepsilon,h}} f(z)\,dz = \pi$$

$$\| $$

$$\int_{\substack{|z|=R \\ z\dots}} - \int_{\substack{|z|=\varepsilon \\ z\dots}} + \int_{\varepsilon+ih}^{R+ih} - \int_{\varepsilon-ih}^{R-ih}$$

$$\int_{|z|=R} \xrightarrow{R\to\infty} 0 \qquad \int_{|z|=\varepsilon} \xrightarrow{\varepsilon\to 0} 0$$

$$f(z) = \frac{\sqrt{z}}{(1+z)^2}$$

$$\lim_{\substack{z\to 0 \\ 0<\operatorname{Arg} z<2\pi}} f(z) = 0$$

$$|f(z)| = \sqrt{|z|}\cdot\frac{1}{|1+z|^2} \xrightarrow{z\to 0} 0$$

$$\lim_{z\to\infty} z f(z) = 0$$

(podle lemmi 1)

$$\Rightarrow$$

$$\pi = \lim_{\substack{\varepsilon\to 0 \\ R\to\infty \\ h\to 0}} \left(\int_{\varepsilon+ih}^{R+ih} f(z) - \int_{\varepsilon-ih}^{R-ih} f(z) \right)$$

Porovnim $\lim \int_{\varepsilon+ih}^{R+ih}$ a $\lim \int_{\varepsilon-ih}^{R-ih}$

$\operatorname{Arg} = \alpha \to 0$ $\qquad$ $\operatorname{Arg} = 2\pi - \alpha \to 2\pi$

$$\lim_{\substack{z\to x \\ \operatorname{Im} z>0}} \sqrt{z} = \underbrace{e^{\pi i}}_{=-1} \cdot \lim_{\substack{z\to x \\ \operatorname{Im} z<0}} \sqrt{z}$$

$z \to x$
$\operatorname{Im} z > 0$

$\operatorname{Arg}\sqrt{z} \to 0$

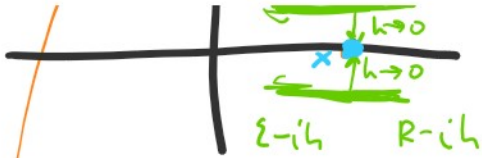
$\sqrt{z} = \sqrt{|z|}\cdot e^{i\cdot\frac{1}{2}\operatorname{Arg} z}$

$\operatorname{Arg}\sqrt{z} = \frac{1}{2}\operatorname{Arg} z$

$\parallel$
-1

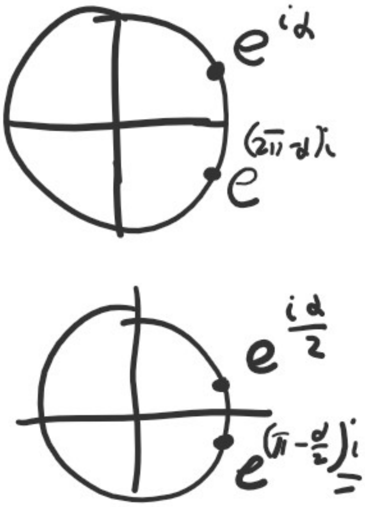
$z \to x$
$\operatorname{Im} z < 0$

$\operatorname{Arg}\sqrt{z} \to \pi$



$$\Rightarrow \lim \int_{\varepsilon+ih}^{R+ih} = -\lim \int_{\varepsilon-ih}^{R-ih}$$

$\sqrt{z}$:



$= e^{\pi i}\cdot e^{-\frac{i\alpha}{2}}$

$\lim\limits_{\alpha\to 0}$

$$\Rightarrow 2\int_0^{+\infty} \frac{\sqrt{x}}{(1+x)^2}\,dx = \pi$$

$$\int_0^{+\infty} \frac{\sqrt{x}}{(1+x)^2}\,dx = \frac{\pi}{2}$$

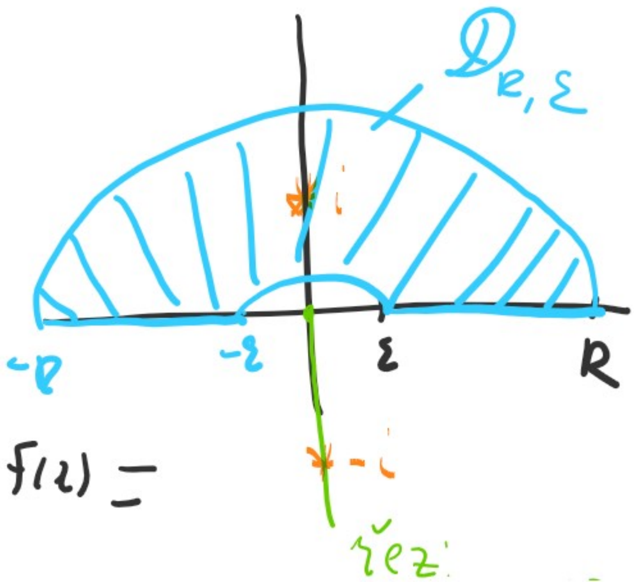
<u>Příklad</u>: $\displaystyle\int_0^{+\infty} \frac{\ln x}{x^2+1}\,dx = ?$

$\parallel$

$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_\varepsilon^R \frac{\ln x}{x^2+1}\,dx$$

Uvažujeme

$$\int_{\partial D} \frac{\ln z\,dz}{1+z^2} = 2\pi i\cdot \operatorname{res}_i f(z) =$$



$$\int_{\partial D_{R,\varepsilon}} \frac{\ln z}{1+z^2} = 2\pi i \cdot \operatorname{res}_i \ldots =$$

řez: $-\frac{\pi}{2} < \operatorname{Arg} z < \frac{3\pi}{2}$

$$= 2\pi i \operatorname{res}_i \frac{\ln(z)}{(z+i)(z-i)} =$$

$f(z) = \frac{\ln z}{1+z^2} \in \mathcal{O}(D_{R,\varepsilon} \setminus \{i\})$

$$= 2\pi i \lim_{z \to i} \frac{\ln z}{z+i} =$$

$$= 2\pi i \frac{\ln i}{2i} = \pi\left(\ln|i| + i\frac{\pi}{2}\right) = \frac{i\pi^2}{2}$$

Ale
$$\int_{\partial D_{R,\varepsilon}} = \int_{\substack{|z|=R \\ \operatorname{Im} z \geq 0}} - \int_{\substack{|z|=\varepsilon \\ \operatorname{Im} z \geq 0}} + \int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}$$

?

$R \to \infty$: 0 $\qquad$ $\varepsilon \to 0$: 0

$$|f(z)| = \frac{|\ln z|}{|1+z^2|} = \mathcal{O}\left(\frac{\ln|z|}{|z|^2}\right)$$

$$\lim_{z \to \infty} z f(z) = 0$$

$$|f(z)| = \mathcal{O}(\ln|z|) = \mathcal{O}(\ln \varepsilon)$$

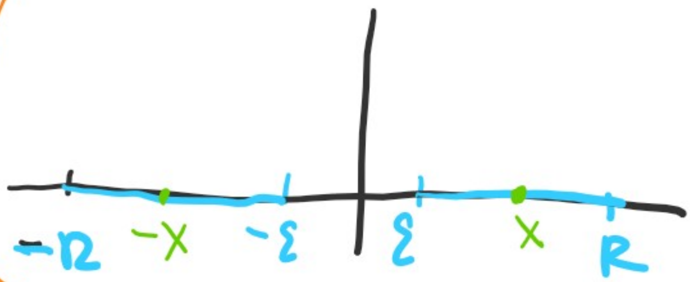
$$\Rightarrow \left| \int_{\substack{|z|=\varepsilon \\ \operatorname{Im} z \geq 0}} \right| \leq \mathcal{O}(\ln|\varepsilon|) \cdot \pi\varepsilon \xrightarrow{\varepsilon \to 0} 0$$

Lemma 1.

Porovním: $\int^{R} \frac{\ln z}{1+z^2} dz$ a $\int^{-\varepsilon} \frac{\ln z}{1+z^2} dz$

Porovnim: $\int_{\varepsilon} \frac{\dots}{1+z^2}\,dz$ a $\int_{-R} \frac{\dots}{1+z^2}\,dz$

$\ln(-x) = \ln x + i\,\mathrm{Arg}(-x) =$
$= \ln x + \pi i$



$$\frac{1}{1+(-x)^2} = \frac{1}{1+x^2}$$

$$\Rightarrow f(-x) = \frac{\ln x + \pi i}{1+x^2} = f(x) + \pi i \cdot \frac{1}{1+x^2} \Rightarrow$$

$$\int_{-R}^{-\varepsilon} \frac{\ln x}{1+x^2} = |x=-y| = \int_{\varepsilon}^{R} \frac{\ln(-y)}{1+y^2}\,dy =$$

$$= \int_{\varepsilon}^{R} \frac{\ln y}{1+y^2} + \pi i \int_{\varepsilon}^{R} \frac{1}{1+y^2}\,dy = \int_{\varepsilon}^{R} \frac{\ln y}{1+y^2} + \pi i\left(\operatorname{arctg} R - \operatorname{arctg} \varepsilon\right)$$

$\Rightarrow$ nahradíme a přimeníme $\lim\limits_{\substack{\varepsilon \to 0 \\ R \to \infty}}$:

$$2\int_0^{+\infty} \frac{\ln x}{1+x^2} + \frac{\pi^2 i}{2} = \frac{i\pi^2}{2} \Rightarrow \boxed{\int_0^{+\infty} \frac{\ln x}{1+x^2}\,dx = 0}$$

Příklad: $\int_0^{+\infty} \frac{\ln x \, dx}{x^2-1} = ?$

$x=1$: funkce spojitá

$x=1$: funkce spojitá

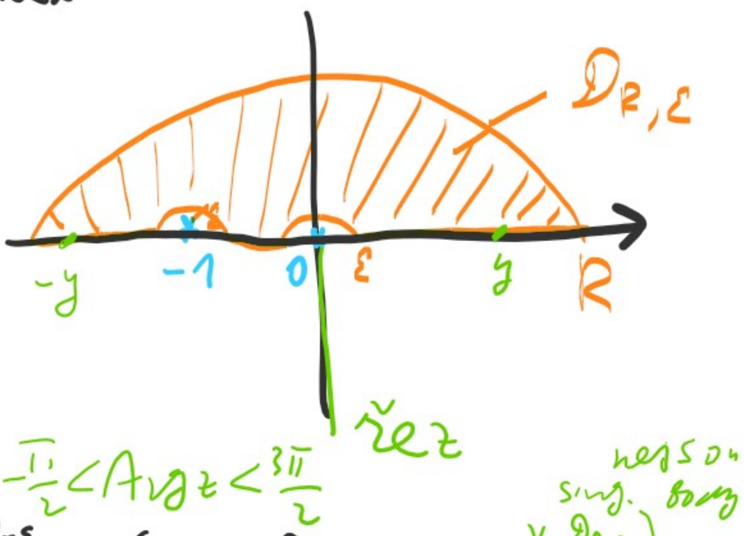
$x=0$: $\sim \ln x$ $\quad\Rightarrow$ converg.

$x=\infty$: $\sim \frac{\ln x}{x^2} < \frac{1}{x\sqrt{x}}$

$$F(z)=\frac{\ln z}{z^2-1}$$

$z=1$ je <u>odstran.</u>

proč $\lim\limits_{z\to 1}\frac{\ln z}{z^2-1}=\frac{1}{2}$



nejsou sing. v $D_{R,\varepsilon}$ bodu

$$\int_{\partial D_{R,\varepsilon}} F(z) = \underbrace{\int_{\substack{|z|=R\\ \operatorname{Im} z\geq 0}}}_{I_1} + \underbrace{\int_{-R}^{-1-\varepsilon}}_{I_2} + \underbrace{\int_{-1+\varepsilon}^{-\varepsilon}}_{I_3} + \underbrace{\int_{\varepsilon}^{R}}_{I_4} - \underbrace{\int_{\substack{|z+1|=\varepsilon\\ \operatorname{Im} z\geq 0}}}_{I_5} - \underbrace{\int_{\substack{|z|<\varepsilon\\ \operatorname{Im} z\geq 0}}}_{I_6} = 0$$

<u>I_1</u>: $|\ln z| = |\ln R + i\overbrace{\operatorname{Arg} z}^{\text{omezen}}| \sim \ln R$

$$\Rightarrow \lim_{\substack{z\to\infty\\ \operatorname{Im} z\geq 0}} zF(z)=0 \Rightarrow I_1 \xrightarrow[R\to\infty]{\text{Lemma 1}} 0$$

<u>I_5</u>: $\lim\limits_{\varepsilon\to 0} I_5 = \left|\underset{\text{z lommuj res}}{\text{Lemma 3}}\right| = \pi i\cdot \operatorname{res} f(z) =$

$\lim F(z)\cdot(z+1) =$

$$=\pi i \lim_{z\to -1}\frac{\ln z}{z-1} = \pi i\,\frac{\ln(-1)}{-2} = -\frac{\pi i}{2}\left(\ln 1 + i\cdot\pi\right) = \frac{\pi^2}{2}$$

I : $|I$ ... $\varepsilon\to 0$

$\underline{I_6}$: $|J_6| \le \pi\varepsilon \cdot \max|f| \le C\cdot\pi\varepsilon\cdot|\ln\varepsilon| \xrightarrow{\varepsilon\to 0} 0$

Teď bereme $\lim\limits_{\substack{R\to\infty \\ \varepsilon\to 0}}$: $0 = 0 + \underbrace{\int_{-\infty}^{0} \frac{\ln z}{z^2-1}\,dz}_{} + \underbrace{\int_{0}^{+\infty} \frac{\ln x}{x^2-1}\,dx}_{\text{naš}} - \frac{\pi^2}{2} + 0$

rozumíme ve smyslu hlav. hodnoty $\lim\limits_{\varepsilon\to 0}\left(\int_{-\infty}^{-1-\varepsilon} + \int_{-1+\varepsilon}^{0}\right)$

$$\int_{-\infty}^{0} \frac{\ln z}{z^2-1}\,dz = \left|z = -y,\ y \ge 0\right| =$$

$$= \int_{0}^{\infty} \frac{\ln(-y)}{y^2-1}\,dy =$$

$$= \int_{0}^{+\infty} \frac{\ln y + i\pi}{y^2-1}\,dy = \underbrace{\int_{0}^{\infty} \frac{\ln y}{y^2-1}\,dy}_{\text{naš}} + \pi i \int_{0}^{\infty} \frac{dy}{y^2-1} =$$

$$= \text{naš} + \pi i \lim_{\varepsilon\to 0}\left(\frac{1}{2}\ln\frac{1+y}{1-y}\Big|_{0}^{1-\varepsilon} + \frac{1}{2}\ln\frac{1+y}{1-y}\Big|_{1+\varepsilon}^{\infty}\right) =$$

$$= |\text{výpočet}| = \text{naš} + 0 \Rightarrow$$

$$2\int_{0}^{\infty} \frac{\ln x\,dx}{x^2-1} - \frac{\pi^2}{2} = 0 \Rightarrow \underline{\underline{\int_{0}^{\infty} \frac{\ln x\,dx}{x^2-1} = \frac{\pi^2}{4}}}$$

Aplikace pro sumace řad

Věta: Nechť $R(z)$ – racion. funkce;

$$R(z)=\frac{P(z)}{Q(z)},\quad \deg Q-\deg P\geq 2$$

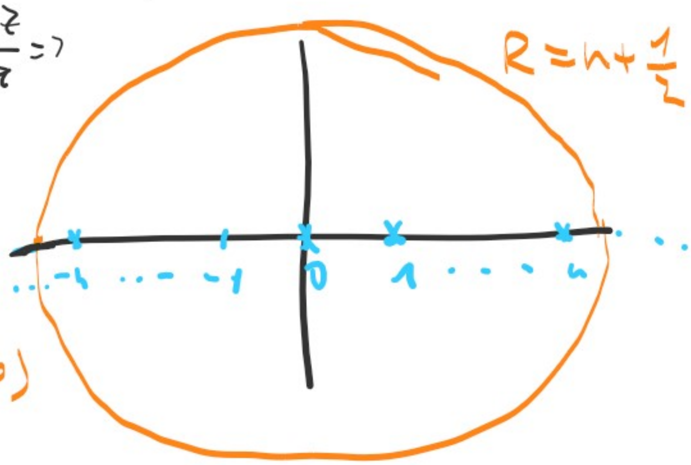
1) Nechť navíc, $Q(n)\neq 0,\ \forall n\in\mathbb{Z} \Rightarrow$

$$\boxed{\sum_{n=-\infty}^{\infty} R(n) = -\pi\sum_{\substack{\text{nulevý}\\ \text{body } Q}} \operatorname{res}\left(R(z)\operatorname{ctg}\pi z\right)}$$

2) Nechť $Q(0)=0$, ale $Q(n)\neq 0$, $n\in\mathbb{Z}, n\neq 0$

$$\Rightarrow \sum_{n\in\mathbb{Z},n\neq 0} R(n) = -\pi \sum_{\substack{\text{nulevý}\\ \text{body } Q,\ z\neq 0}} \operatorname{res}\left(R(z)\operatorname{ctg}\pi z\right) - \pi \operatorname{res}_0\left(R(z)\operatorname{ctg}\pi z\right)$$

Důkaz; $\xrightarrow{\quad} 1)$ $\operatorname{ctg}\pi z=\frac{\cos\pi z}{\sin\pi z}=?$



sing body $\operatorname{ctg}\pi z$: $n\in\mathbb{Z}$

(pro velký n, R není sing. na kružnice)

Fakt (bez důkazu) $|\operatorname{ctg}\pi z|\leq M$ – const

$$\forall z, \forall R = n + \tfrac{1}{2}$$

Ted: uvaž. $\displaystyle\int_{|z|=R} R(z)\,\mathrm{ctg}\,\pi z \, dz =$

$$= |\text{Rez věta}| = 2\pi i \sum_{\substack{\text{sing. bodů} \\ \text{v } B_R}} \mathrm{Res}\left(\mathrm{ctg}\,\pi z \, R(z)\right) =$$

$$= 2\pi i \left(\sum_{k=-n}^{n} \mathrm{res}_k \left(\mathrm{ctg}\,\pi z \, R(z) \right) \right) + 2\pi i \left(\sum_{\substack{\text{nuly} \\ Q}} \mathrm{res}\left(R(z)\,\mathrm{ctg}\,\pi z\right) \right)$$

$$\int_{|z|=R} R(z)\,\mathrm{ctg}\,\pi z \xrightarrow{R \to \infty} 0$$

$$\left(|\mathrm{ctg}\,\pi z| \le M \Rightarrow \text{Lemma 1 platí} \right)$$

$$z R(z) \xrightarrow{z \to \infty} 0$$

$$\Rightarrow \text{dereme } \lim_{\substack{R \to \infty \\ (n \to \infty)}} : \quad \underbrace{\sum_{-\infty}^{\infty} \mathrm{res}_k \, R(z)\,\mathrm{ctg}\,\pi z}_{R(k)/\pi} = -\sum_{\substack{\text{nul.} \\ Q}} \mathrm{res}\left(R(z)\,\mathrm{ctg}\,\pi z\right)$$

$$\mathrm{res}_k \, R(z)\,\mathrm{ctg}\,\pi z = |z = k + w| = \mathrm{res}_0 \, R(k+w)\,\mathrm{ctg}\,\pi w =$$

$$= \mathrm{res}_0 \, R(k+w) \frac{\overset{w \to 0}{\cos \pi w}\to 1}{\sin \pi w} = \left| \begin{array}{l} w = 0 \text{ je} \\ \text{pól řádu } 1 \\ \text{nebo odstran.} \\ \text{když } R(k) = 0 \end{array} \right| =$$

$$= \lim_{w \to 0} R(k+w) \frac{\cos \pi w}{\sin \pi w} \cdot w = \frac{1}{\pi} \cdot R(k) \cdot 1 \Rightarrow$$

$$\lim \frac{\sin z}{z} = 1$$

$\lim \frac{\sin z}{z} = 1$

$$\sum_{-\infty}^{+\infty} R(k) = -\pi \sum_{\text{nul. } Q} \operatorname{res}\left(R(z)\operatorname{cotg}\pi z\right).$$

Případ 2) sleduje z důkazu.

<u>Příklad</u>: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = ?$

$R(z) = \frac{1}{z^2}$; $\Rightarrow$ podle věty!

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^2} = 2\sum_{1}^{\infty}\frac{1}{n^2} = -\pi \operatorname{res}_0 \frac{1}{z^2}\operatorname{cotg}\pi z =$$

$$= -\pi \cdot \operatorname{res}_0 \frac{\cos\pi z}{z^2 \sin\pi z} = |\text{pól řádu} = 3| =$$

$$= -\pi \operatorname{res}_0 \left(\frac{1}{z^2}\left(1 - \frac{\pi^2 z^2}{2} + \dots\right)\frac{1}{\pi z}\,\frac{1}{1 - \frac{z^2\pi^2}{6} + \dots}\right) =$$

$\sin x = x - \frac{x^3}{6} + \dots$

$$= -\cdot\operatorname{res}_0\left(\frac{1}{z^3}\left(1 - \frac{\pi^2 z^2}{2} + \dots\right)\left(1 + \frac{z^2\pi^2}{6} + \dots\right)\right) =$$

$\frac{1}{1-y} = 1 + y + y^2 + \dots$

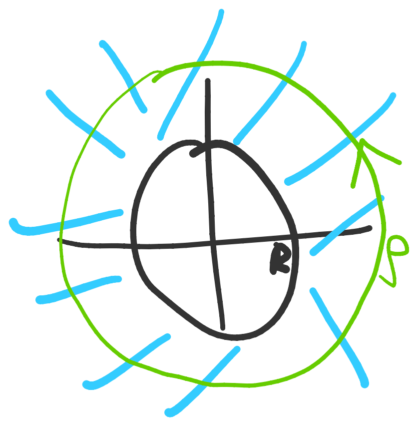
$$= -\cdot\left(-\frac{\pi^2}{2} + \frac{\pi^2}{6}\right) = +\frac{\pi^2}{3} \Rightarrow$$

$\sum^{\infty} \frac{1}{\ldots} \ldots \pi^2$

$$\boxed{\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}}$$

Podobně je možné vypočíst $\zeta(2k) = \sum_{n=1}^\infty \frac{1}{n^{2k}}$

Reziduum v ∞



$f \in \mathcal{O}(\mathbb{C} \setminus \overline{B_R(0)})$

∞ – isol. sing. bod

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

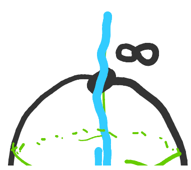
<u>Def</u>: $\boxed{\operatorname{res}_\infty f(z) := -c_{-1}}$

Proč tak?

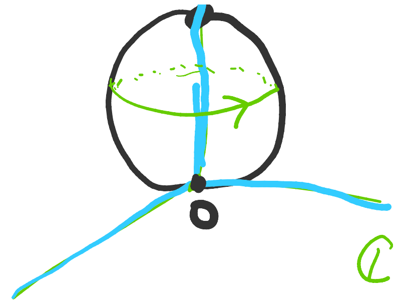
hvražnjeme $\int\limits_{\substack{|z|=\rho \\ \rho>R}} f(z) = \int\limits_{|z|=\rho} \sum_{n=-\infty}^{\infty} c_n z^n =$

$= \left| \text{conv. stejn. na } |z|=\rho \right| = \sum_{n=-\infty}^{\infty} c_n \int z^n dz = \left| \int z^n = \begin{cases} 0, & n \neq -1 \\ 2\pi i \end{cases} \right|$

$= 2\pi i \cdot c_{-1}$;



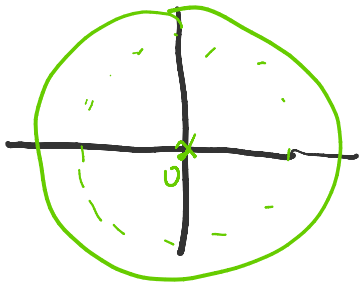
ale „posit" směr pro ∞ =



= „negat" pro 0

$$\Rightarrow \boxed{\oint_{\partial B_{\frac{1}{\varrho}}(\infty)} f(z) = 2\pi i \cdot \operatorname{res}_\infty f}$$

díváme z ∞!



Příklad: $\displaystyle\int_{|z|=100} \frac{dz}{z-z^{100}} = 2\pi i \sum_{j=1}^{100} \operatorname{res}_{a_j} f(z)$ — špatný nápad

$z - z^{100} = 0 \Rightarrow z = 0,\ z = \sqrt[99]{1}$

Lepší: uvažovat ∞ jako sing bod!

V $B_{\frac{1}{\varrho}}(\infty)$: $\displaystyle \frac{1}{z-z^{100}} = \frac{-1}{z^{100}} \cdot \frac{1}{1-\frac{1}{z^{99}}} =$

$$= -\frac{1}{z^{100}}\left(1 + \frac{1}{z^{99}} + \dots\right)$$

$$\Rightarrow \operatorname{res}_\infty f(z) = 0$$

$$\Rightarrow \int_{|z|=100} f(z)\,dz = 0.$$

Věta: Nechť $R(z) = \dfrac{P(z)}{Q(z)}$ - racion funkce;

$\Rightarrow$

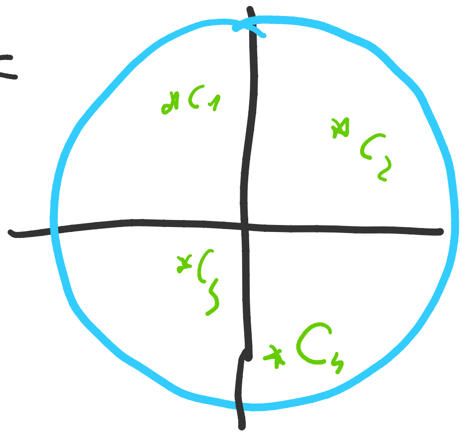
$$\boxed{\sum_{\substack{\text{sing.}\\ \text{body } R \text{ v } \overline{\mathbb{C}}}} \operatorname{res} R(z) = 0 \quad (!)}$$

<u>Důkaz</u>: nechť $c_1, \dots, c_n$ – nul. body Q
$c_0 = \infty$

$\Rightarrow \{c_0, c_1, \dots, c_n\}$ – isol. sing. body R;
v ostat. bodech R je hol.

Uvažujme $B_r(0) \supset \{c_1, \dots, c_n\}$

$$\frac{1}{2\pi i}\int_{|z|=r} R(z)\,dz = |\text{Rez. věty}| =$$



$$= \sum_{j=1}^{n} \operatorname{res}_{c_j} R(z)$$

Z druhé strany: $\frac{1}{2\pi i}\int_{|z|=r} R(z)\,dz = -\operatorname{res}_{\infty} R(z)$

$$\Rightarrow \sum_{j=1}^{n} \operatorname{res}_{c_j} R(z) + \operatorname{res}_{\infty} R(z) = 0.$$

$\blacksquare$

Harmonické funkce

Harmonické funkce

$\leftarrow$ R-změna

Def: nechť $u(x,y)$ je funkce regularity C^2 v oblasti $D \subset \mathbb{R}^2$ ($D \subset \mathbb{C}$). $u(x,y)$ se nazývá harmonická když:

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv 0$$

Notace: $Harm(D)$ – prostor harm funkce v D.

Δ – Laplaceův operátor

Příklad: $\ell(x,y) = ax + by + c \in Harm(\mathbb{R}^2)$

$$x^2 - y^2 \in Harm(\mathbb{R}^2)$$

$$\frac{x}{x^2+y^2} \in Harm(\mathbb{R}^2 \setminus \{0,0\})$$

(výpočet)

Věta: Nechť $F = u + iv \in \mathcal{O}(D)$.

Pak $u, v \in Harm(D)$

$(\mathrm{Re}\,F,\ \mathrm{Im}\,F \in Harm(D))$

$(\operatorname{Re} f, \operatorname{Im} f \in \operatorname{Harm}(D))$

<u>Důkaz</u>: $\begin{cases} \frac{\partial}{\partial x}\; u_x = v_y \;\frac{\partial}{\partial y} \\ \frac{\partial}{\partial y}\; u_y = -v_x \;\frac{\partial}{\partial x} \end{cases}$ – Cauchy-Riemann

$\begin{cases} u_{xx} = v_{xy} \\ + \\ u_{yy} = -v_{xy} \end{cases}$ $\quad (f \in C^\infty(D) \Rightarrow u, v \in C^\infty(D)$
a můžeme říci že $v_{xy} = v_{yx}$

$\Rightarrow \boxed{u_{xx} + u_{yy} = 0}$ $\qquad \blacksquare$

$\begin{cases} u_{xy} = v_{yy} \\ u_{xy} = -v_{xx} \end{cases} \Rightarrow \boxed{v_{xx} + v_{yy} = 0}$

<u>Def</u>: když $f \in O(D)$, $f = u + iv \Rightarrow$

v se nazývá <u>harmonicky sdružená</u> funkce pro u.

$-if = v - iu \Rightarrow -u$ – harm. sdruž. pro v.
$\in O(D)$

<u>Věta</u>: $u \in \operatorname{Harm}(D) \Rightarrow \exists$ max 1

harm. sdr. funkce v v D až do konstanty
(sčítání konstanty)

<u>Důkaz</u>: $v \exists \Rightarrow \begin{cases} v_x = -u_y \\ v_y = u_x \end{cases} \Rightarrow v_x, v_y$ – známe

$\Rightarrow$ známe $v(x,y)$ až do konstanty.

$\{\text{sdr. funkce}\} = V_0 + C, \ C \in \mathbb{R}$

konkrétní sdr. funkce.

<u>Poznámka:</u> $Harm(D)$ – lin. prostor nad $\mathbb{R}$!,

protože Δ – lin. operátor

$\alpha_1 u_1 + \alpha_2 u_2 \in Harm(D)$

<u>Příklady:</u> $f = \frac{1}{z}$, $f = \frac{\bar{z}}{z\bar{z}} = \frac{x - iy}{x^2+y^2} \Rightarrow$

$u = \frac{x}{x^2+y^2}$, $-v = \frac{y}{x^2+y^2} \in Harm(\mathbb{C}\setminus\{0\})$

u, v – harm. sdruž.

$f = \operatorname{Ln} z = \ln|z| + i \operatorname{Arg} z$

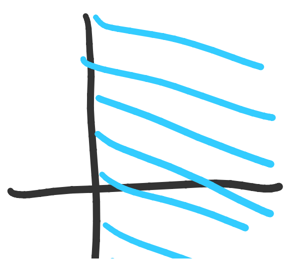
jednoznačný

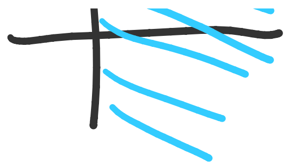
$\Rightarrow \ln|z| \in Harm(\mathbb{C}\setminus\{0\})$ – Tady sdruž. $\nexists$

$\operatorname{Arg} z \in Harm(\mathbb{C} \setminus \text{řez})$.

Např., pro $D = \{\operatorname{Re} z > 0\}$

$\arg z = \operatorname{arctg}\frac{y}{x} \in Harm(D)$



$\arg z = \operatorname{arctg} \frac{y}{x} \in \mathrm{Harm}(D)$ 

<u>Výsledek</u>: harm.-sdruž. funkce může být mnohoznačná.

<u>Kdy je sdruž. funkce jednozn.?</u>

<u>Věta</u>: nechť D - jednod. souv. oblast;
$\Rightarrow \forall u \in \mathrm{Harm}(D)$, $\exists$ sdruž. funkce $v \in \mathrm{Harm}(D)$
(zejména, $\exists f \in \mathcal{O}(D)$: $u = \mathrm{Re} f$).

<u>Důkaz</u>: chceme najít v: $\begin{cases} v_x = -u_y \\ v_y = u_x \end{cases}$

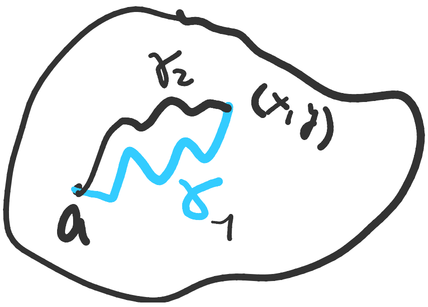
(potřeb. v se zadanými parc. deriv!) — stand. úloha: hledáme v: $\begin{cases} \frac{\partial v}{\partial x} = P \\ \frac{\partial v}{\partial y} = Q \end{cases}$

v - potenciál vekt. pole (P, Q)

(Analýza) v jednod. souv. oblasti, potenciál $\exists$, $V = \int_a^{(x,y)} P\,dx + Q\,dy$, $a \in D$

(!)

křivk. int. podél γ



<u>Pokud</u> platí podmínky

rovna p-ati podmínky (a $^{-1}$

Cauchy: $\boxed{\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}}$

(jiný zápis: $dV = \overset{=\omega}{P\,dx + Q\,dy}$)

$$d\omega = 0 \Rightarrow \omega = dV$$

Lemma Poincaré

V naše situace: $P = -u_y,\ Q = u_x$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \iff -u_{yy} = u_{xx} \iff \Delta u = 0$$

pravda

$$\Rightarrow V\ \exists;\quad \exists f = u + iV \in \mathcal{O}(D)$$

$\blacksquare$

Důsledek: D - jednod. souv. oblast $\Rightarrow$

$$u \in Harm(D) \iff u = \operatorname{Re} f,\ f \in \mathcal{O}(D)$$

Neplatí pro obecný oblast!

$$u(z) = \ln|z| \neq \operatorname{Re} f(z) \text{ v } \mathbb{C}\setminus\{0\}$$

$u(z) = \ln|z|$ $\nexists$ ke $f(z)$ v $\mathbb{C}\setminus\{0\}$

(jinak $f = \operatorname{Ln} z$)
není

Příklad: $u = xy$; najít v, f v $\mathbb{C}$.

$$\begin{cases} v_x = -u_y \\ v_y = u_x \end{cases} \Rightarrow \quad v_x = -x \Rightarrow v = -\frac{1}{2}x^2 + a(y)$$

Pak: $v_y = u_x \Rightarrow a'(y) = y \Rightarrow a(y) = \frac{y^2}{2} + \beta, \beta \in \mathbb{R}$.

$$\Rightarrow v = -\frac{1}{2}x^2 + \frac{1}{2}y^2 + \beta$$

$$f = xy + i\left(-\frac{1}{2}x^2 + \frac{1}{2}y^2 + \beta\right) =$$

$$= -\frac{i}{2}\left(x^2 - y^2 + 2xy\cdot i\right) + i\beta = -\frac{i}{2}z^2 + i\beta, \quad \beta \in \mathbb{R}$$

Příklad: $v = \frac{y}{x^2+y^2}$, $u, f = ?$ $\quad D = \mathbb{C}\setminus\{0\}$

$$\begin{cases} u_x = v_y = \frac{x^2-y^2}{(x^2+y^2)^2} \\ u_y = -v_x = \frac{2xy}{(x^2+y^2)^2} \end{cases} \Rightarrow u = \int \frac{2xy\,dy}{(x^2+y^2)^2} =$$

dif. podle y!

$$= x\int \frac{d(x^2+y^2)}{(x^2+y^2)^2} = -\frac{x}{x^2+y^2} + a(x),$$

$\int \frac{dt}{t^2} = -\frac{1}{t} + c$

$\int \frac{dt}{t^2} = -\frac{1}{t} + c$

$$u_x = \frac{x^2-y^2}{(x^2+y^2)^2}; \quad \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} + a'(x)$$

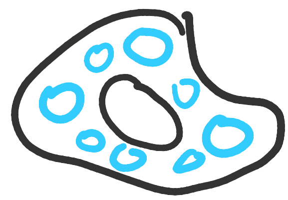
$$\Rightarrow a'(x) = 0, \quad a(x) = a \in \mathbb{R}$$

$$\Rightarrow u = -\frac{x}{x^2+y^2} + a$$

$$f = -\frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2} + a = -\frac{\bar{z}}{z\bar{z}} + a = -\frac{1}{z} + a.$$

<u>Dusledok Vety</u>: $u \in Harm(\mathcal{D}) \Rightarrow$

$\forall B_r(a) \subset \mathcal{D}, \exists f \in \mathcal{O}(B_r(a))$: $\operatorname{Re} f = u$.



<u>Vlastnosti harm. funkce</u>

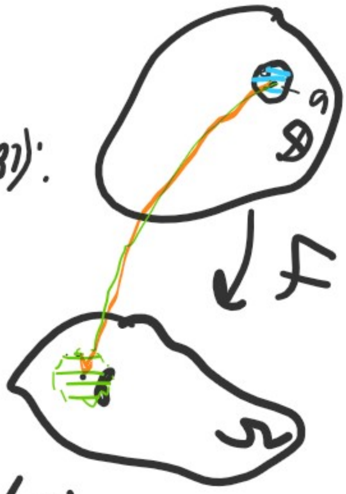
<u>Veta 1</u>: $f: \mathcal{D} \xrightarrow{hol} \Omega$ $(f(\mathcal{D}) \subset \Omega)$,

a $u \in Harm(\Omega) \Rightarrow$

$u(f(z)) = u \circ f \in Harm(\mathcal{D})$.

<u>Dokaz</u>: to staci dokazat že $\forall a \in \mathcal{D}, \exists B_r(a) \subset \mathcal{D}$:

$u\circ f \in Harm(B_r(a))$



$b = f(a)$, v $B_R(b)$, $\exists g \in O(B_R(b))$:

$u = \mathrm{Re}\, g$ (důsledek);

Bereme $B_r(a)$: $f(B_r(a)) \subset B_R(a)$

$\Rightarrow$ v $B_r(a)$ uvažujeme $g\circ f \in O(B_r(a))$;

v $B_r(a)$ platí: $u\circ f = \mathrm{Re}\, \underbrace{g\circ f}_{hol} \Rightarrow$

$u\circ f \in Harm(B_r(a)) \Rightarrow u\circ f \in Harm(D)$

∎

<u>Příklad</u>: když $f \in O(D)$, $f \neq 0$ v D

$$\Rightarrow \boxed{\ln|f| \in Harm(D)}$$

<u>Věta 2</u>: $u \in Harm(D) \Rightarrow u \in C^\infty(D)$!

Navíc, u je <u>analytická</u>:

$\forall B_r(a) \subset D$, $a = \alpha+\beta i$, $u(x,y) = \sum_{k,n=0}^{\infty} a_{kn} (x-\alpha)^k (y-\beta)^n$

$k,n=0$

$$= \sum_{k=0}^{\infty} \frac{1}{2}\left(c_k(z-a)^k + \overline{c_k}\,(\bar{z}-\bar{a})^k\right)$$

c_i libov. — moc speciální — Není libovolný!

<u>Důkaz</u>: pro C^∞-vlastnost stačí dokázat, že $u \in C^\infty(B_r(a))$, $\forall B_r(a) \subset D$

Bereme $B_r(a) \subset D$; $\exists f: u = \operatorname{Re} f$,

$f \in \mathcal{O}(B_r(a)) \Rightarrow f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$

$f \in C^\infty$ zřejmě $\Rightarrow u = \operatorname{Re} f \in C^\infty(B_r(a))$

$$u = \operatorname{Re} f = \frac{1}{2}(f + \bar{f}) = \sum_{n=0}^{\infty} \frac{1}{2} c_n (z-a)^n + \frac{1}{2}\overline{c_n}(\bar{z}-\bar{a})^n.$$

$\blacksquare$

<u>Poznámka</u>: V zápisu $u(x,y) = \sum_{n=0}^{\infty} P_n(x-\alpha, y-\beta)$,

P_n – homog. polynom od 2 prom. stupně $n \Rightarrow$

$P_n \in \operatorname{Harm}(\mathbb{C})$.

Řada konverguje stejnoměrně na komp. podmnožinách kruhu.

<u>Věta 3</u>: Nechť $f \in \mathcal{O}(D)$; $\overline{B_r(a)} \subset D$. Pak:

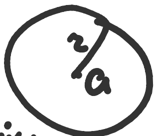
(i) $f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})\,dt$ — střední hodnota

(a) $f(a)=\frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt$ — střední hodnota f na kružnici $\partial B_r(a)$.

(b) $f(a)=\frac{1}{\pi r^2}\int_{B_r(a)} f(z)\,dx\,dy$ — střední hodnota f v kruhu $B_r(a)$

Důkaz: (a) V zorec Cauchy $f(a)=\frac{1}{2\pi i}\int_{|\xi-a|=r}\frac{f(\xi)\,d\xi}{\xi-a}$

Nachradíme $\xi=a+re^{it}\Rightarrow$



$$f(a)=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(a+re^{it})}{re^{it}}\cdot i r e^{it}\,dt=\frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt$$

(b) Píšeme v $B_r(a)$ Tayl řadu:

$$f(z)=\sum_{n=0}^{\infty}c_n(z-a)^n$$ — konv. stejnom v $\overline{B_r(a)}$

$\Rightarrow$ integr:
$$\int_{B_r(a)} f(z)\,dx\,dy=\sum_{n=0}^{\infty}c_n\int_{B_r(a)}(z-a)^n\,dx\,dy=$$

$$=\left|\begin{array}{l}\text{Polární}\\ \text{nahrazení:}\end{array}\ \begin{array}{l}x=\rho\cos\varphi+\alpha\\ y=\rho\sin\varphi+\beta,\quad a=\alpha+\beta i\\ z-a=\rho(\cos\varphi+i\sin\varphi)=\rho e^{i\varphi}\end{array}\ \begin{array}{l}0\le\rho\le r\\ 0\le\varphi\le 2\pi\\ Jac=\rho\end{array}\right|=$$

$$=\sum_{n=0}^{\infty}c_n\cdot\int_0^{2\pi}\int_0^{r}\rho^n e^{in\varphi}\cdot\rho\,d\rho\,d\varphi=\sum_{n=0}^{\infty}c_n\cdot\int_0^{r}\rho^{n+1}\,d\rho\cdot\int_0^{2\pi}e^{in\varphi}\,d\varphi=$$

$$=c_0\cdot\frac{r^2}{2}\cdot 2\pi+\underbrace{\sum_{n=1}^{\infty}c_n\cdot(\text{něco})\cdot\frac{1}{in}e^{in\varphi}\Big|_0^{2\pi}}_{\text{“}0\text{”}}=$$

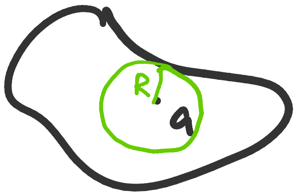
$$=\pi r^2 c\ldots$$

$$\overset{!!}{0}$$

$$= \pi r^2 c_0 \Rightarrow f(a) = c_0 = \frac{1}{\pi r^2} \int_{B_r(a)} f(z)\,dx\,dy. \qquad \blacksquare$$

<u>Poznámka</u>: ve části (B), můžeme

$r \to R = \operatorname{dist}(a, \partial D) \Rightarrow f(a) = \dfrac{1}{\pi R^2} \displaystyle\int_{B_R(a)} f(z)\,dx\,dy$



Pokud v $B_r(a)$, $u = \operatorname{Re} f \in \operatorname{Harm}(D)$, máme důsledek:

<u>Věta o střední hodnotě</u>: $u \in \operatorname{Harm}(D)$, $\forall\, \overline{B_r(a)} \subset D$, platí:

(a) $u(a) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} u(a + re^{it})\,dt$ — střední hodnota na kružnici

(B) $u(a) = \dfrac{1}{\pi r^2} \displaystyle\int_{B_r(a)} u(x,y)\,dx\,dy$ — střední hodnota ve kruhu

(taky můžeme vzít $r = R = \operatorname{dist}(a, \partial D)$).