Laurentovy řady

Def: Laurentovy řad je řad ve tvaru $\sum_{n=-\infty}^{\infty} c_n (z-a)^n$, $c_n \in \mathbb{C}$, $a \in \mathbb{C}$

centr

Popíšeme jako: $$\underbrace{\sum_{n=0}^{\infty} c_n (z-a)^n}_{\text{regulárna část}} + \underbrace{\sum_{n=-\infty}^{-1} c_n (z-a)^n}_{\text{hlavní část}}$$

regulárna část (-fce Tayl. řada / mocnina řada)

hlavní část (singulárna část)

Řad konvergje v bodu z_0, když regul. a hlav. část konvergnji (a součet = součet$_1$ + součet$_2$).

Oblast konvergencie: vnitřní část množiny kde Laurentovu řad konvergje.

množině kde Laurentova řad konverguje.

Co to může být?

$$\frac{1}{z-a} := t \Rightarrow \underbrace{\sum_{n=0}^{\infty} c_n (z-a)^n}_{\text{moc. řada}} + \underbrace{\sum_{k=1}^{\infty} c_{-k} \cdot t^k}_{\text{moc. řada}}$$

Věta Cauchy-Adamara $\Rightarrow$

Oblast konvergence:

$$\begin{cases} |z-a| < R, & 0 \leq R \leq \infty \\ |t| < \frac{1}{r} \Leftrightarrow |z-a| > r, & 0 \leq r \leq \infty \end{cases}$$

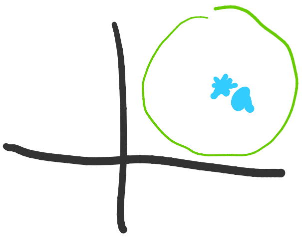
$t = \frac{1}{z-a}$

$\Rightarrow$ konečně: $\boxed{r < |z-a| < R}$

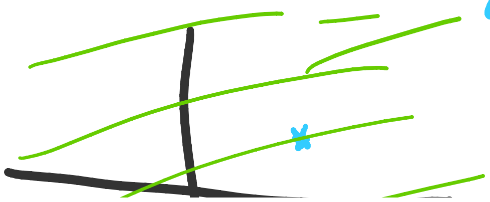
obecně, to je mezikruží

Přesnější:

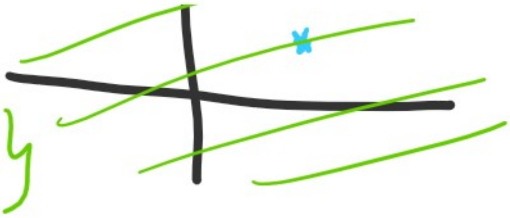
1) $r = 0$, $R \neq 0$ $\Rightarrow$ $\{0 < |z-a| < R\}$, může nebýt jen $B_R'(a)$



$B_R'(a)$, $\begin{cases} \text{konečný } R < \infty \\ \text{nekonečný } R = \infty \end{cases}$

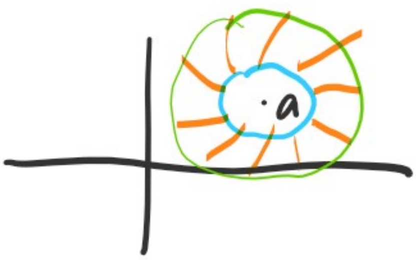


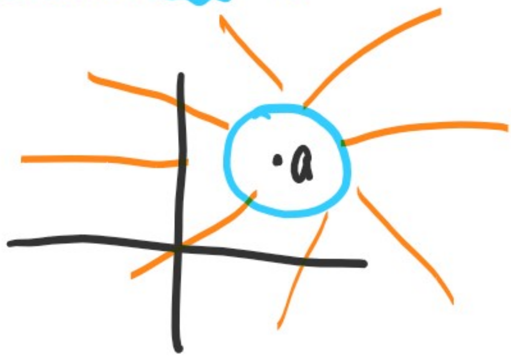
$\mathbb{C} \setminus \{a\}$



2) $\emptyset$ $\quad r \geq R$

3) $0 < r < R$ $\quad$ mezikruh $\begin{cases} \text{konečné } R < \infty \\ \text{nekonečné } R = \infty \end{cases}$





Příklady

$a = 0$

1) $$\sum_{n=-\infty}^{\infty} z^n = \sum_{n=0}^{\infty} z^n + \sum_{n=-\infty}^{-1} z^n$$

$|z| < 1$

$$\sum_{n=-\infty}^{-1} z^n = \sum_{1}^{\infty} \left(\frac{1}{z}\right)^n$$

$\left|\frac{1}{z}\right| < 1 \iff |z| > 1$

$\Rightarrow \emptyset$

2) $$\sum^{\infty} z^n = \frac{1}{\;} + \sum^{\infty} z^n \qquad |z| < 1$$

2) $\sum_{n=-1} z^n = \frac{1}{z} + \sum_{0} z^n$ $\quad \begin{cases} |z| < 1 \\ z \neq 0 \end{cases}$

$B_1'(0)$

3) $\sum_{0}^{\infty} z^n$ $\quad \{|z| < 1\} = B_1(0)$

4) $\sum_{n=-\infty}^{\infty} \frac{z^n}{2^{|n|}} = \sum_{n=0}^{\infty} \frac{z^n}{2^n} + \sum_{n=-\infty}^{1} \frac{z^n}{2^{-n}} =$

$= \sum_{n=0}^{\infty} \frac{z^n}{2^n} + \sum_{1}^{\infty} \left(\frac{1}{z}\right)^k \frac{1}{2^k}$

$|z| < 2$ $\quad \left|\frac{1}{z}\right| < 2 \iff |z| > \frac{1}{2}$

$\left\{\frac{1}{2} < |z| < 2\right\}$

4) $\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{z^3}\cdot\frac{1}{3!} + \frac{1}{z^5}\cdot\frac{1}{5!} - \ldots$

$\sin t = t - \frac{t^3}{3!} t \ldots$ – konv. $\forall t \Rightarrow$

$t = \frac{1}{z}$

$\Rightarrow \frac{1}{z} \in \mathbb{C} \Rightarrow \{0 < |z| < \infty\}$

$\mathbb{C} \setminus \{0\}$

$\mathbb{C} \setminus \{0\}$

Hl. Věta: Nechť $f \in \mathcal{O}(\overbrace{\{r<|z-a|<R\}}^{D})$

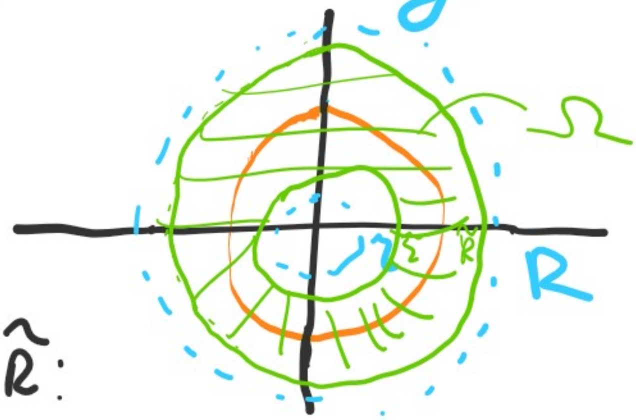
$r<R$, $0 \le r < \infty$, $0 < R \le \infty \implies$

f je představována v D jako součet Laurentovy řady:

$$f(z)=\sum_{n=-\infty}^{\infty} c_n (z-a)^n, \quad \text{kde}$$

$$\boxed{c_n = \frac{1}{2\pi i}\int_{|z-a|=\rho} \frac{f(\xi)\,d\xi}{(\xi-a)^{n+1}}}$$ – vzorec Cauchy

$r<\rho<R$



Důkaz: $(a=0)$

uvažujeme $\forall \tilde r, \rho, \tilde R$:

$r<\tilde r<\rho<\tilde R<R$

Teď, přiměníme integ vzorec Cauchy pro oblast $\Omega=\{\tilde r<|z-a|<\tilde R\}$

(2-souv oblast, $f \in \mathcal{O}(\overline{\Omega})$)

(Ω-souv oblast, $f \in \mathcal{O}(\Omega)$)

$$f(\underset{\Omega}{\underset{\cap}{z}}) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\xi)\,d\xi}{\xi - z} = |\text{podle def}| =$$

$$= \underbrace{\frac{1}{2\pi i} \int_{|\xi| = \hat{R}} \frac{f(\xi)\,d\xi}{\xi - z}}_{\parallel \atop F_1} - \underbrace{\frac{1}{2\pi i} \int_{|\xi| = \hat{\varrho}} \frac{f(\xi)\,d\xi}{\xi - z}}_{F_2} =$$

{ upravná kopie důkazu Tayl. rozvoju }

$$F_1(z) = \frac{1}{2\pi i} \int_{|\xi| = \hat{R}} \frac{f(\xi)\,d\xi}{(\xi - a) - (z - a)} = \frac{1}{2\pi i} \int_{|\xi| = \hat{R}} f(\xi)\,d\xi \cdot \frac{1}{\xi - a}\, \frac{1}{1 - \frac{z-a}{\xi - a}}$$

$$= \left| \begin{array}{c} |z-a| < |\xi - a| \\ \left|\frac{z-a}{\xi-a}\right| < 1 \end{array} \right| = \frac{1}{2\pi i} \int_{|\xi| = \hat{R}} \frac{f(\xi)\,d\xi}{\xi - a} \sum_{n=0}^{\infty} \left(\frac{z-a}{\xi - a}\right)^n =$$

= |podle stejnom konv.| =

= součet integrálu $= (z-a)^n \cdot c_n$,

$$c_n = \frac{1}{2\pi i} \int \frac{f(\xi)\,d\xi}{(\xi - a)^{n+1}}$$

$$F_2(z) = \frac{1}{2\pi i} \int_{|\xi - a| = \hat{\varrho}} \frac{f(\xi)\,d\xi}{\xi - z} = \frac{1}{2\pi i} \int \frac{f(\xi)\,d\xi}{(\xi - a) - (z - a)} =$$

$$\ldots \frac{1}{2\pi i} \int\limits_{|\xi-a|=\tilde{r}} \ldots \frac{\ldots}{1-t} \ldots \frac{1}{2\pi i} \int \frac{\ldots}{(\xi-a)-(z-a)} =$$

$$= -\frac{1}{2\pi i} \int \frac{F(\xi)\,d\xi}{(z-a)\left(1-\frac{\xi-a}{z-a}\right)} = \left| \begin{array}{l} |\xi-a| < |z-a| \\ \left|\frac{\xi-a}{z-a}\right| < 1 \end{array} \right| =$$

$$= -\frac{1}{2\pi i} \int \frac{F(\xi)\,d\xi}{z-a} \sum_{k=0}^{\infty} \left(\frac{\xi-a}{z-a}\right)^k =$$

$$= \sum_{k=0}^{\infty} (z-a)^{-k-1} \cdot B_k, \quad B_k = -\frac{1}{2\pi i} \int F(\xi) \cdot (\xi-a)^k$$

$$-k-1 := n \Rightarrow \sum_{n=-\infty}^{-1} B_{-n-1} \cdot (z-a)^n$$

$$B_{-n-1} = -\frac{1}{2\pi i} \int \frac{F(\xi)\,d\xi}{(\xi-a)^{n+1}}$$

$$F(z) = F_1(z) - F_2(z)$$

když označíme $c_n := -B_{n-1}$

a porovníme, a nahradíme do $F_1, F_2 \Rightarrow$ dostaneme přesně

$$F(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

$n=-\infty$

$$C_n = \frac{1}{2\pi i} \int\limits_{|\zeta-a|=\tilde{R}} \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n+1}} \quad (n \geq 0)$$

$$C_n = \frac{1}{2\pi i} \int\limits_{|\zeta-a|=\tilde{r}} \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n+1}} \quad (n < 0)$$

ale podle věty Cauchy pro oblast $\{\rho < |z-a| < \tilde{R}\}$ a funkce $\frac{f(z)}{(z-a)^{n+1}}$, máme: $\int\limits_{|\zeta-a|=\tilde{R}} \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n+1}} - \int\limits_{|\zeta-a|=\rho} \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n+1}} = 0$

$$\Rightarrow \int\limits_{|\zeta-a|=\rho} = \int\limits_{|\zeta-a|=\tilde{R}}$$

Stejně pro $n<0$ $\int\limits_{|\zeta-a|=\tilde{r}} = \int\limits_{|\zeta-a|=\rho}$,

a to dokáza vzorci Cauchy. $\blacksquare$

<u>Důsledek</u>: $f \in \mathcal{O}\{r < |z-a| < R\}$

platí nerovnice Cauchy pro

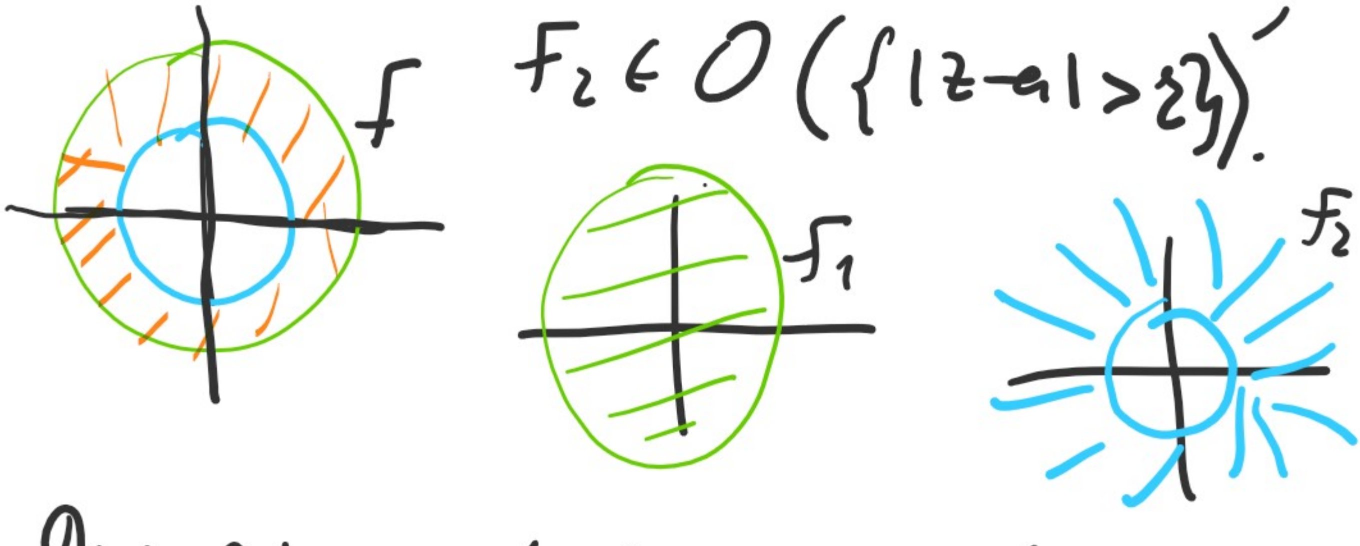
platí nerovnice Cauchy pro koef. Laur. řady:

$$|c_n| \le \frac{M}{\rho^n}, \quad r < \rho < R, \quad M = \max_{|z-a|=\rho} |f|$$ — nerovnice Cauchy

Důkaz: $c_n = \frac{1}{2\pi i} \int\limits_{|\zeta-a|=\rho} \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n+1}} \Rightarrow$

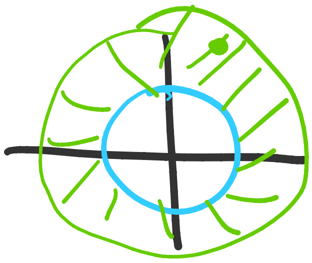
$$\Rightarrow |c_n| \le \frac{1}{2\pi} \cdot 2\pi\rho \cdot \frac{1}{\rho^{n+1}} \cdot M = \frac{M}{\rho^n}$$

Důsledek: $f \in \mathcal{O}(\{r < |z-a| < R\})$ může být představována jako $f_1(z) + f_2(z)$, $f_1 \in \mathcal{O}(\{|z-a| < R\})$, $f_2 \in \mathcal{O}(\{|z-a| > r\})$.



Důkaz: regul. část má poloměr konverg. $\ge R$ (to je mocn. řada

konverg. $\geq R$ (to je mocn. řada
která konverguje
v bodech z, $|z| \to R$)



$\Rightarrow f_1$ — regul. část

stejně, hlav. část $\sum_{-\infty}^{-1} c_n (z-a)^n = \{|z-a| = t\} =$

$= \sum_{k=1}^{\infty} c_{-k} t^k$; konverguj v bodech

t: $|t| \to \frac{1}{2} \Rightarrow$ poloměr konv. $\geq \frac{1}{2}$

$|z-a| > 2 \iff |t| < \frac{1}{2}$

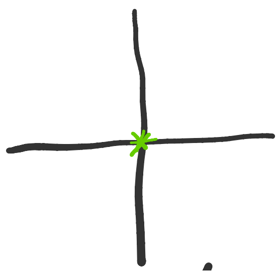
$\Rightarrow$ konv. v $\{2 < |z-a| < \infty\} \Rightarrow$
je tam hol — f_2

Příklady

Najít Laurentův rozvoj $f(z)$ ve všech možných (maxim.) oblastí:

1) $f(z) = (z+1)e^{\frac{1}{2z}}$

$f \in O(\mathbb{C} \setminus \{0\}) \Rightarrow$



má tam Laur. řadu, a to je max oblast

$$f(z)=(z+1)\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{2^n}\frac{1}{z^n} = z+\frac{3}{2}+\sum_{n=1}^{\infty} c_n\cdot\frac{1}{z^n}$$

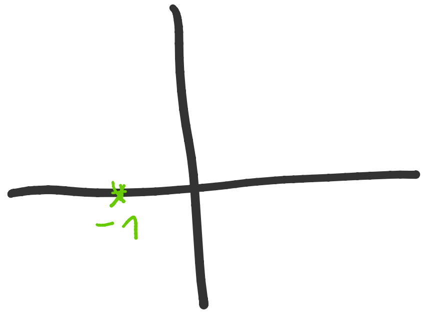
$$(z+1)\cdot\left(1+\frac{1}{2z}+\dots\frac{1}{n!\,2^n}\frac{1}{z^n}+\frac{1}{(n+1)!\,2^{n+1}z^{n+1}}+\dots\right)$$

$$c_n=\frac{1}{n!\,2^n}+\frac{1}{(n+1)!\,2^{n+1}}=\frac{2n+3}{(n+1)!\cdot 2^n}$$

2) $$f(z)=\frac{z^2+1}{z+1}=\frac{z^2+z-z-1+2}{z+1}=$$

$$=z-1+\frac{2}{z+1}$$

$f\in O(\mathbb{C}\setminus\{-1\})$



Chled. rozvoj v $(\mathbb{C}\setminus\{-1\})$

$$\sum_{n=-\infty}^{\infty} c_n(z+1)^n \;—\; ?$$

$z+1=t$

$z=t-1$

$$f=\underbrace{t-2}_{\text{reg. část}}+\underbrace{\frac{2}{t}}_{\text{hl. část}}=(z+1)-2+\frac{2}{z+1}$$

3) $$f=\frac{1}{z^2-2z}=\frac{1}{z(z-2)}=\frac{A}{z}+\frac{B}{z-2}$$

$$\frac{1}{z^2-2z} = \frac{1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}$$

$$1 = A(z-2) + Bz; \quad B = \frac{1}{2}; \quad A = -\frac{1}{2}$$

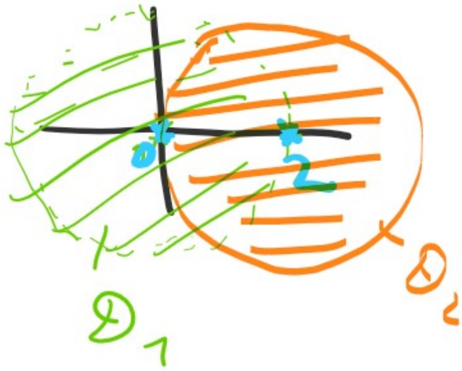
$$F = \frac{1/2}{z} - \frac{1/2}{z-2}$$

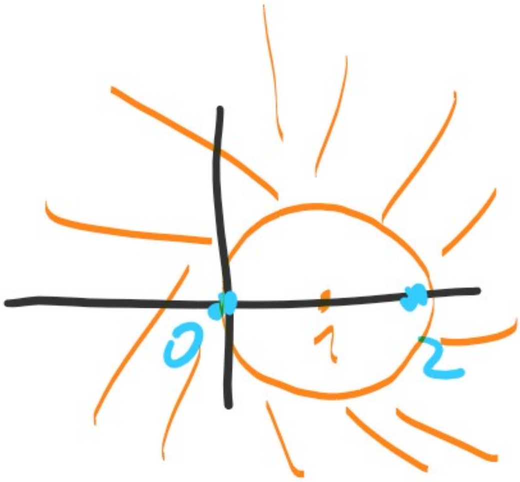
3 max oblasti:

1) $D_1 = \{0 < |z| < 2\}$

2) $D_2 = \{0 < |z-2| < 2\}$

3) $D_3 = \{1 < |z-1| < \infty\}$





V D_1: $0 < |z| < 2$; $a = 0$

$$F = \frac{1}{2}\frac{1}{z} - \frac{1}{2}\frac{1}{z-2} = \frac{1}{2z} + \frac{1}{4}\frac{1}{1-\frac{z}{2}} =$$

$$= \frac{1}{2z} + \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \frac{1}{2z} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+2}}$$

V D_2: $a = \overset{\to t=0}{2} \Rightarrow z-2 = t, \; z = 2+t$

$$F = \frac{1}{2z} - \frac{1}{2}\frac{1}{z-2} = \frac{1}{2(\ldots)} - \frac{1}{\ldots} -$$

$$f = \frac{1}{2z} - \frac{1}{2}\,\frac{1}{z-2} = \frac{1}{2(t+2)} - \frac{1}{2t} =$$

$$= -\frac{1}{2t} + \frac{1}{4}\,\frac{1}{1+\frac{t}{2}} = -\frac{1}{2t} + \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{t}{2}\right)^n =$$

$-\left(-\frac{t}{2}\right)$

$$= |z = t+2| = -\frac{1}{2(z-2)} + \sum_{n=0}^{\infty} \frac{(z-2)^n \cdot (-1)^n}{2^{n+2}}$$

<u>v D_3:</u> $|z-1| > 1$, $\frac{1}{z-1} = t$,

$z - 1 = \frac{1}{t}$, $z = 1 + \frac{1}{t} = \frac{t+1}{t}$

$|t| < 1$

$z - 2 = \frac{1}{t} - 1 = \frac{1-t}{t}$

$$\Rightarrow F(z) = \frac{1}{2}\cdot\frac{t}{t+1} - \frac{1}{2}\,\frac{t}{1-t} =$$

$$= \frac{1}{2}\,\frac{-2t^2}{(1+t)(1-t)} = -t^2\cdot\frac{1}{1-t^2} =$$

$|t|<1$, $|t^2|<1$

$$= -t^2\cdot\sum_{n=0}^{\infty} t^{2n} = -\sum_{n=0}^{\infty} t^{2n+2} = -\sum_{n=0}^{\infty}\frac{1}{(z-1)^{2n+2}}$$

4) $f(z) = \dfrac{z^2+1}{z^3-4z}$ $\rightarrow$ $z_1 = 0$, $z_2 = 2$, $z_3 = -2$

$z_3 = -2$

najít Laurentův rozvoj z centrem $z_0 = 1$, ve všech možných max mezikruh

$$\boxed{f(z) = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+2}} = \frac{z^2+1}{z(z-2)(z+2)}$$

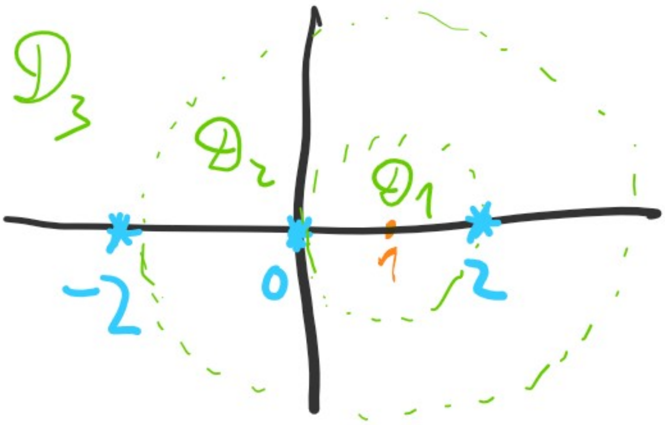
• $z(z-2)(z+2)$, a přes $\left\{ \begin{array}{l} z=0 \\ z=2 \\ z=-2 \end{array} \right.$ najdeme

A, B, C.

$\mathcal{D}_1: \quad |z-1| < 1$

$\mathcal{D}_2: \quad 1 < |z-1| < 3$

$\mathcal{D}_3: \quad |z-1| > 3$



$z - 1 = t, \; z = 1 + t \Rightarrow$

$$f = \frac{A}{t+1} + \frac{B}{t-1} + \frac{C}{t-3}, \quad t_0 = 0$$

$\mathcal{D}_1: \quad |t| < 1$

$\mathcal{D}_2: \quad 1 < |t| < 3$

$\mathcal{D}_3: \quad 3 < |t| < \infty$

$\mathcal{D}_3: 3 < |t| < \infty$

$\underline{\mathcal{D}_1}$: $\frac{A}{t+1} = A(1 - t + t^2 - t^3 + \dots) = A \sum_{n=0}^{\infty} t^n \cdot (-1)^n$

$$\frac{B}{t-1} = -\frac{B}{1-t} = -B \sum_{n=0}^{\infty} t^n$$

$$\frac{C}{t-3} = -\frac{C}{3} \frac{1}{1-\frac{t}{3}} = -C \sum_{n=0}^{\infty} \frac{t^n}{3^{n+1}}$$

$|\,| < \frac{1}{3}$

$$f = f_1 + f_2 + f_3$$

$1 < |t| < 3$

$\underline{\mathcal{D}_2}$: $\frac{A}{t+1} = \frac{A/t}{1+\frac{1}{t}} = \frac{A}{t} \sum_{n=0}^{\infty} \frac{(-1)^n}{t^n} =$ $\quad n = -1-k$, $k = -1-n$

$\left|\frac{1}{t}\right| < 1$

$$= A \sum_{k=-\infty}^{-1} t^k \cdot (-1)^{k-1}$$ Lanz.

$$\frac{B}{t-1} = \frac{B/t}{1-\frac{1}{t}} = \frac{B}{t} \sum_{n=0}^{\infty} \frac{1}{t^n} = B \sum_{k=-\infty}^{-1} t^k$$ Lanz.

$$\frac{B}{t-3} = -\frac{B}{3} \frac{1}{1-\frac{t}{3}} = -\frac{B}{3} \sum_{n=0}^{\infty} \frac{t^n}{3^n} = -B \sum_{n=0}^{\infty} \frac{t^n}{3^{n+1}}$$

Taylor

$F = f_1 + f_2 + f_3$

Tayl.

D_3: $|t| > 3$;

$$\frac{A}{t+1} = \frac{A}{t}\,\frac{1}{1+\frac{1}{t}} \stackrel{\uparrow}{=} A\sum_{-\infty}^{-1} t^k (-1)^{k-1}$$

$$\frac{B}{t-1} = \frac{B}{t}\,\frac{1}{1-\frac{1}{t}} \stackrel{\uparrow}{=} B\sum_{-\infty}^{-1} t^k$$

$$\frac{C}{t-3} = \frac{C}{t}\,\frac{1}{\underbrace{1-\frac{3}{t}}_{|\,|<1}} = \frac{C}{t}\sum_{n=0}^{\infty}\frac{3^n}{t^n} = |k=-1-n| =$$

$$= C\sum_{-\infty}^{-1} t^k \cdot 3^{-1-k}$$

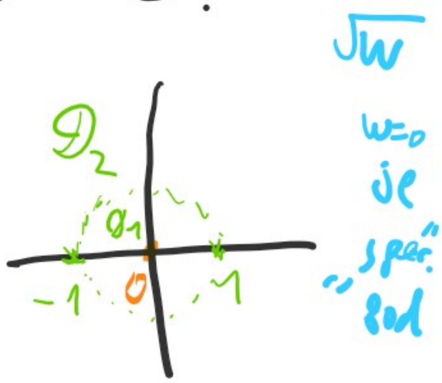
$F = f_1 + f_2 + f_3$

konečný výsledek: v D_1, D_2, D_3, popis všech Laur. koef.

popis všech Laur. koef.

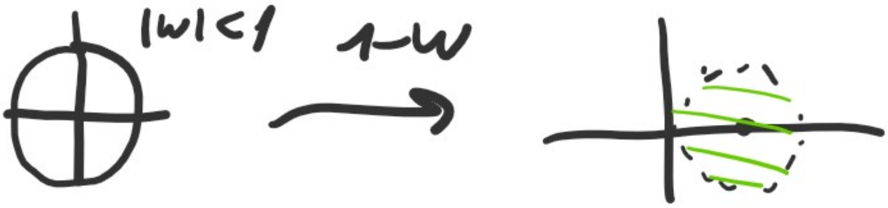
5) Pro mnohozn. funkce f, najít Laur rozvoj její regul. větví v max možných oblastí, $z_0 = 0$.

$$f = \sqrt{z^2 - 1}$$



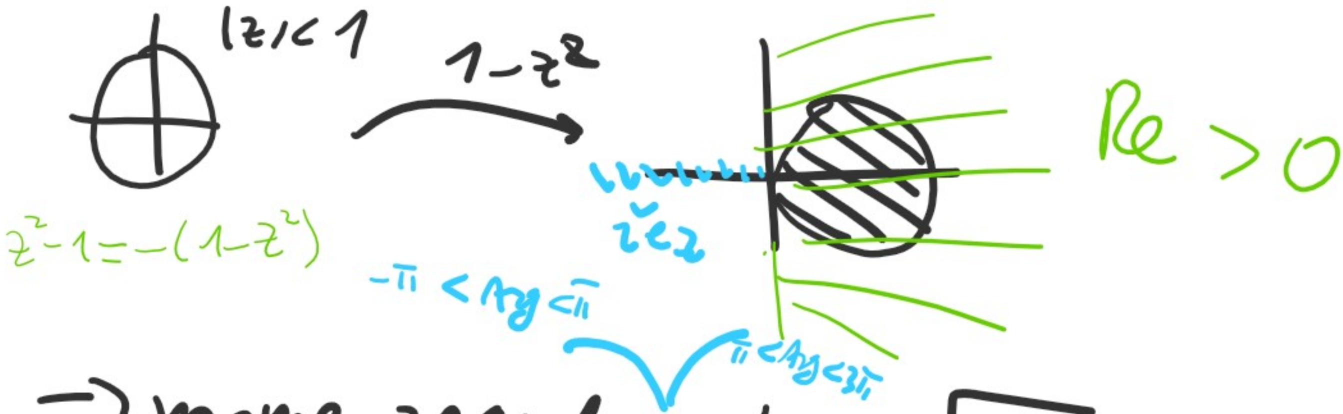
$z^2 - 1 = 0$; $z_{1,2} = \pm 1$

$|z| < 1$, $|z|^2 < 1 \Rightarrow 1 - |z|^2 \in (0,1)$



$\mathrm{Re}(1-z^2) > 0$ pro kruh $|z|<1$

$\Rightarrow \exists$ regul. větev $\sqrt{1-z^2}$



$\Rightarrow$ máme regul. větev $\sqrt{1-z^2}$ jako složenou

$|z|<1$

inne stożenij

$|z|<1$

$$\Rightarrow F = i\cdot\sqrt{1-z^2} = \left| z^2 = t,\ |t|<1 \right| =$$

$$= i\cdot\sqrt{1-t} = i\cdot(\pm 1)\cdot\Big(1+\tfrac{1}{2}(-t) +$$

$\boxed{(1+z)^\alpha}$

$$+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}t^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(-t)^3 + \dots$$

$$= \mp i \sum_{n=0}^{\infty} z^{2n}\cdot\frac{(2n-3)!!}{n!\cdot 2^n}$$

— 2 větvi f v $B_1(0)$.

D_2: $|z|>1$; $\sqrt{z^2-1} = \sqrt{z^2(1-\frac{1}{z^2})} =$

$$= z\cdot\sqrt{1-\frac{1}{z^2}} = \Big\{ |z|>1,\ \left|\tfrac{1}{z}\right|<1,$$

$$1-\frac{1}{z^2}\in\{\operatorname{Re} w>0\}\Big\} = \{\exists\ 2\ \text{regulární větvi}\} =$$

$\boxed{(1+z)^\alpha}$

$$= \pm z\cdot\Big(1-\frac{1}{z^2}\frac{1}{2}+\frac{1}{z^4}\cdot\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} - \frac{1}{z^6}\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} +$$

$$+\dots\Big) = \mp z\cdot\sum\frac{1}{z^{2n}}\frac{(2n-1)!!}{\dots} -$$

$$\ldots] = \mp z \cdot \sum_{n=0} \frac{1}{z^{2n}} \frac{(2n-1)!!}{2^n\, n!} =$$

6) $f = \operatorname{Ln}\left(\frac{2+z}{2-z}\right)$, stejná úloha $(z_0 = 0)$

„problémní" body: $z_1 = 2$, $z_2 = -2$

$$f = \operatorname{Ln}(2+z) - \operatorname{Ln}(2-z) =$$

$$= \cancel{\operatorname{Ln} 2} + \operatorname{Ln}\left(1+\frac{z}{2}\right) - \operatorname{Ln}\left(1-\frac{z}{2}\right) \cancel{-\operatorname{Ln} 2} =$$

$$= \left| \begin{array}{l} \left|\frac{z}{2}\right| < 1 \Rightarrow \operatorname{Re}\left(1 \pm \frac{z}{2}\right) > 0 \Rightarrow \exists \text{ regul. větví} \\ \text{jako složení; } \operatorname{Ln} w,\ -\pi + 2\pi k < \operatorname{Arg} w < \pi + 2\pi k) \end{array} \right| =$$

$$= 2\pi k i + \ln\left(1+\frac{z}{2}\right) - 2\pi l i - \ln\left(1-\frac{z}{2}\right) =$$

$$= 2\pi n i + \left(\frac{z}{2} - \frac{z^2}{4}\cdot\frac{1}{2} + \frac{z^3}{8}\cdot\frac{1}{3} - \ldots\right) -$$

$\boxed{\ln(1+z)}$

$z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots$

$$- \left(-\frac{z}{2} - \frac{z^2}{4}\cdot\frac{1}{2} - \frac{z^3}{8}\cdot\frac{1}{3} - \ldots\right) =$$

$$= 2\pi n i + z + 2\cdot\frac{z^3}{8}\cdot\frac{1}{3} + 2\cdot\frac{z^5}{2^5}\cdot\frac{1}{5} + \ldots =$$

$$= 2\pi n i + \sum_{k=1}^{\infty} z^{2k-1} \cdot \frac{1}{2^{2k-2}} \cdot \frac{1}{2k-1}$$

$\Rightarrow |z| > 1$

$\sum\limits_{k=1} \quad \dfrac{\cdot}{2^{2k-2}}\cdot\dfrac{1}{2k-1}$

$|z|>1$
$\left|\frac{2}{z}\right|<1$

$\underline{D_2}$: $f=\ln\left(\frac{2+z}{2-z}\right)=\ln(2+z)-\ln(2-z)=$

$(-1)\cdot\left(1-\frac{2}{z}\right)$

$=\cancel{\ln z}+\ln\left(1+\frac{2}{z}\right)-\cancel{\ln z}-\ln\left(\frac{2}{z}-1\right)=$

$(2\pi k i)=+\ln\left(1+\frac{2}{z}\right)-\pi i-\ln\left(1-\frac{2}{z}\right)=\left|\left|\frac{2}{z}\right|<1\right|=$

$=2\pi k i-\pi i+\left(\frac{2}{z}-\frac{4}{z^2}\cdot\frac{1}{2}+\frac{8}{z^3}\cdot\frac{1}{3}-\dots\right)+$

$+\left(\frac{2}{z}+\frac{4}{z^2}\cdot\frac{1}{2}+\frac{8}{z^3}\cdot\frac{1}{3}+\dots\right)=$

$=\pi i(2k-1)+\sum\limits_{n=1}^{\infty}\frac{1}{z^{2n-1}}\cdot\frac{2^{2n}}{2n-1}$

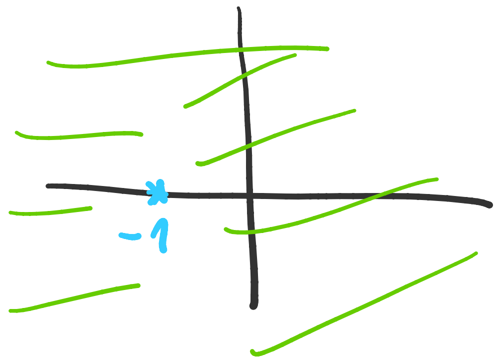
7) $F=z\sin\frac{z}{z+1}$; $z_0=-1$

[illegible]: $\boxed{C_{-3}}$

$D:\{0<|z+1|<\infty\}$

$z+1=t$; $z=t-1$

$0<|t|<\infty$



$F=(t+1)\cdot\sin\left(\frac{t-1}{t}\right)=(t+1)\sin\left(1-\frac{1}{t}\right)=$

$=(t+1)\left(\sin 1\cos\frac{1}{t}-\cos 1\sin\frac{1}{t}\right)=$

$$(\qquad \qquad \qquad t) =$$

$$=(t+1)\left(\sin 1\left(1-\frac{1}{t^2}\cdot\frac{1}{2!}+\frac{1}{t^4}\cdot\frac{1}{4!}-\dots\right)-\right.$$
$$\left.-\cos 1\left(\frac{1}{t}-\frac{1}{t^3}\cdot\frac{1}{3!}+\frac{1}{t^5}\cdot\frac{1}{5!}-\dots\right)\right)$$

<u>Chceme t^{-3}</u>: $\left[\begin{array}{l} 1\cdot t^{-3} \\ t\cdot t^{-4}\end{array}\right. \Rightarrow$

$$C_{-3}=\frac{\cos 1}{3!}+\frac{\sin 1}{4!}$$

8) $F=\dfrac{\cos\left(\frac{1}{\sqrt{z}}\right)}{1+z}$; $\quad C_{-3}=?\quad z_0=0$

(a dokázat že Laur. rozvoj $\exists!$)

Fakt, F <u>není</u> mnohoznačná $\to$??

$$\cos\frac{1}{\sqrt{z}}=1-\frac{1}{z}\cdot\frac{1}{2!}+\frac{1}{z^2}\cdot\frac{1}{4!}-\frac{1}{z^3}\cdot\frac{1}{6!}+\dots$$

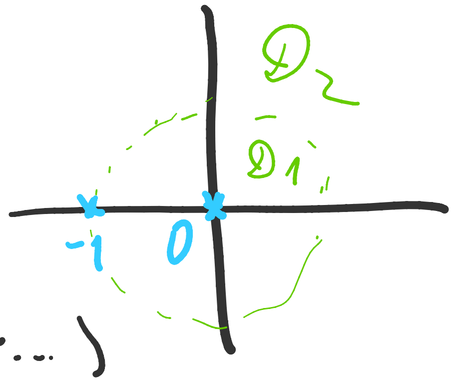
$\frac{1}{z}=t \Rightarrow$ řada converg. $\forall t$ ($R=\infty$) lehko dokázat

$\Rightarrow f(t)\in\mathcal{O}(\mathbb{C}) \Rightarrow f(z)\in\mathcal{O}(\mathbb{C}\setminus\{0\})$

$\Rightarrow$

| D$_?$

$\Rightarrow$



$\underline{\mathcal{D}_1}$: $f = \left(1 - \frac{1}{z}\cdot\frac{1}{2!} + \frac{1}{z^2}\frac{1}{4!} - \frac{1}{z^3}\cdot\frac{1}{6!} + \dots\right)\cdot\left(1 - z + z^2 - z^3 + \dots\right)$

$\Rightarrow C_{-3} = -\frac{1}{6!} - \frac{1}{8!} - \frac{1}{10!} - \dots =$

$= -\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{8!} + \frac{1}{10!} + \dots\right) + 1 + \frac{1}{2} + \frac{1}{24} =$

$= \frac{37}{24} - \operatorname{ch} 1$

$\operatorname{ch} z = \frac{e^z + e^{-z}}{2} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots$

g) $\sum\limits_{n=1}^{\infty} \frac{n}{z^n} = ?$ $\quad (|z| > 1)$

$\frac{1}{z} = t \Rightarrow |t| < 1$; $\quad \sum\limits_{n=1}^{\infty} n t^n$ – converg. $|t| < 1$

$R = 1$

$f = t \cdot \sum\limits_{n=1}^{\infty} n t^{n-1} = t \cdot \left(\sum\limits_{1}^{\infty} t^n\right)' =$

$t \cdot \left(\frac{t}{t+1}\right)' = t \cdot \frac{1}{(1+t)^2}$

$$t \cdot \left(\frac{t}{t+1}\right) = t \cdot \frac{1}{(1+t)^2}.$$