Question G, H Lie groups

$\psi : \mathfrak{g} \longrightarrow \mathfrak{h}$ Lie algebra homo. Does there exist Lie group homom.

$\varphi : G \longrightarrow H$ s.t. $\varphi' = \psi$?

Thm. 1.48 Let G and H be Lie groups and let $\psi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism between their Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$.

Then $\exists$ a local homomorphism $\varphi : \underset{\subseteq G}{U} \longrightarrow H$ s.t. $\varphi' = \psi$.

If G is simply connected, then $\exists$ a homomorphism of Lie groups $\varphi : G \longrightarrow H$ s.t. $\varphi' = \psi$.

<u>Proof</u> Consider the Lie group $G \times H$ and recall that its Lie algebra is direct $\mathfrak{g} \oplus \mathfrak{h}$

Set $\mathfrak{k} := \{ (X, \varphi(X)) : X \in \mathfrak{g} \} \subseteq \mathfrak{g} \oplus \mathfrak{h}$ graph of φ.

- $\mathfrak{k}$ is a linear subspace, since φ is linear.
- It is also a subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$, since φ is a Lie algebra homom.:

$$[(X, \varphi(X)), (Y, \varphi(Y))] = (\underbrace{[X,Y]}_{\in \mathfrak{g}}, \underbrace{[\varphi(X), \varphi(Y)]}_{= \varphi([X,Y])}) \in \mathfrak{k}.$$

By Thm. 1.33, $\exists$ a virtual Lie subgroup $i: K \longrightarrow G \times H$ with Lie algebra isomorphic to $\underline{\mathfrak{k}} \subseteq \mathfrak{g} \oplus \mathfrak{h}$.

- $pr_1 : G \times H \longrightarrow G$
  $pr_2 : G \times H \longrightarrow H$ are Lie group homomorphism.

$\Longrightarrow$ $\underline{\pi} := pr_1 \circ i : K \longrightarrow G$ is a Lie group homomorphism.

$\pi' : \mathfrak{k} \longrightarrow \mathfrak{g}$ $\quad \pi'(X, \psi(X)) = X$ is a linear isomorphism.

By ③ of Thm. 1.46, π is a covering map and a a local diffeomorphism from K to G_0:

$\pi|_V : V \xrightarrow{\sim} U$ diffeom. between open neighbhds V and U of $e \in K$ in K resp. $e \in G$ in G.

Define $\varphi : U \longrightarrow H$ by $\varphi := \underset{\sim}{pr_2} \circ \underline{\underline{(\pi|_V)^{-1}}} : U \longrightarrow H$.

- φ is a local homeomorphism, since $(\pi|_V)^{-1}$ is a homeomorphism.
- By construction, $\varphi' : x \longmapsto (x, \varphi(x)) \longrightarrow \varphi(x)$, i.e. $\varphi' = \varphi$.

If G is simply connected, $G = G_0$ and $\pi : K \to G$ is a homeomorphism and a local diffeomorphism.

$\Rightarrow$ π is an isomorphism of Lie groups

and $\varphi := pr_2 \circ \pi^{-1}$ is the required map.

$\square$.

We recall again some notions from algebraic topology:

X topological space

If X satisfies certain connectedness assumptions (always satisfied for manifolds), then $\exists$ a covering map

$$p : \tilde{X} \longrightarrow X$$

with $\tilde{X}$ a simply connected topological space.

It is called the universal cover of X. (it is unique up to isomorphism).

Properties

(1) If $q : Y \to X$ is another covering map and Y is path connected, then $\exists$ a covering map $\tilde{p} : \tilde{X} \to Y$ s.t. $p = q \circ \tilde{p} : \tilde{X} \to X$.

$$\begin{tikzcd} \tilde{X} \arrow[r, "p"] \arrow[d, dashed, "\tilde{p}"'] & X \\ Y \arrow[ur, "q"'] & \end{tikzcd}$$

(Universal cover covers all connected covers of X).

$\Longrightarrow$ Universal cover of X is unique.

(2) Lifting property: Y simply connected, $f : Y \to X$ continuous, then $\exists$ a continuous map $\tilde{f} : Y \to \tilde{X}$ s.t.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

commutes.

f is uniquely determined by its value in a point.

Suppose $p : N \to M$ is a covering of a smooth manifold M. with at most countably many sheets (for $U \subseteq M$ with $p^{-1}(U) = \bigcup_{i \in I} V_i$, I is countable).

$\Rightarrow$ N is necessarily Hausdorff and AA2, since M is.

Moreover, let (U, u) be a chart of M s.t. $p^{-1}(U) = \bigcup_{i \in I} V_i$ for open subsets V_i of N.

$\Rightarrow$ $u \circ p|_{V_i} : V_i \to u(U)$ is a chart for N.

$\Rightarrow$ N is a smooth mfd. and $p : N \to M$ is smooth and $p|_{V_i}$ is a diffeom. $\forall i \in I$.

Thm. 1.49 Let $\mathfrak{g}$ be a finite-dim. Lie algebra.

Then there exist a unique (up to isomorphism) simply connected Lie group $\tilde{G}$ with Lie algebra $\mathfrak{g}$. Any other Lie group G with Lie algebra (isomorphic to) $\mathfrak{g}$ is isomorphic to the quotient of $\tilde{G}$ by a discrete normal subgroup $H \subseteq \tilde{G}$, which is contained in the center $Z(\tilde{G})$ of $\tilde{G}$.

Proof By Thm. 1.35 (Lie's 3rd fund. Theorem), $\exists$ a connected Lie group G with Lie algebra $\mathfrak{g}$. Let $p: \tilde{G} \to G$ be the universal cover of G. Then $\tilde{G}$ can be made into a smooth mfd. s.t. p is a local diffeomorphism.

$\tilde{G}$ has a Lie group structure :

Fix $\tilde{e} \in \tilde{G}$ s.t. $p(\tilde{e}) = e \in G$.

$\mu \circ p \times p : \tilde{G} \times \tilde{G} \longrightarrow G \times G \longrightarrow G$ is a continuous map.

$\Longrightarrow \exists !$ continuous lift $\tilde{\mu} : \underline{\tilde{G} \times \tilde{G} \longrightarrow \tilde{G}}$ s.t.

$\tilde{\mu}(\tilde{e}, \tilde{e}) = \tilde{e}$.

Similarly, $\exists !$ lift $\tilde{\nu} : \tilde{G} \longrightarrow \tilde{G}$ of $\nu \circ p : \tilde{G} \longrightarrow G$

s.t. $\tilde{\nu}(\tilde{e}) = \tilde{e}$.

By restricting to open subsets, smoothness of μ, ν and p implies smoothness for $\tilde{\mu}$ and $\tilde{\nu}$.

Remains to show that $(\tilde{\mu}, \tilde{\nu}, \tilde{e})$ satisfies the group axioms:

Associativity: $\tilde{\mu} \times (id \times \tilde{\mu})$ and $(\tilde{\mu} \circ (\tilde{\mu} \times id))$ lift the same map $\tilde{G} \times \tilde{G} \times \tilde{G} \longrightarrow G$ and have the same value at $(\tilde{e}, \tilde{e}, \tilde{e})$. $\Longrightarrow$ they coincide, i.e. $\tilde{\mu}$ defines an associative multiplication.

Similarly, one verifies the other group axioms.

$\longrightarrow$ $(\tilde{G}, \tilde{\mu}, \tilde{\nu})$ is a Lie group. (with Lie algebra isomorphic to $\mathfrak{g}$. $p' : \tilde{\mathfrak{g}} \xrightarrow{\sim} \mathfrak{g}$.)

Suppose $\hat{G}$ is another connected Lie group with Lie alg. $\mathfrak{g}$.

By Thm. 1.48, $\exists$ a Lie group homomorphism $\varphi : \tilde{G} \to \hat{G}$ s.t. $\varphi' = \mathrm{id}_{\mathfrak{g}}$.

By Thm. 1.46, φ is a covering map (and a local diffeom.) which induces an isomorphism $\tilde{G}/\ker(\varphi) \simeq \hat{G}$.

$\ker(\varphi)$: discrete normal subgroup contained in the center of $\tilde{G}$.

If $\hat{G}$ is simply connected, then the local diffeom. φ must be a diffeom., and hence an isomorphism of Lie groups. $\square$

By Thm. 1.49, the map that associates to a Lie group its Lie algebra induces a bijection

$$\left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of simply connected} \\ \text{Lie groups} \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of real finite-dim.} \\ \text{Lie algebras} \end{array} \right\}$$

Similarly, for complex Lie groups and Lie algebras.

Classifying connected Lie groups boils down to the classification of Lie algebras.

1.6 Remarks on the classification of Lie algebras

$\mathfrak{g}$ Lie algebra

Then its derived series is inductively defined as follows:

$$\mathfrak{g}^{(1)} := \mathfrak{g} \qquad \mathfrak{g}^{(k+1)} := [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$$

$\rightsquigarrow$ $\mathfrak{g} \supseteq \mathfrak{g}^{(2)} \supseteq \mathfrak{g}^{(3)} \supseteq \dots \supseteq \mathfrak{g}^{(k)} \supseteq \dots$

ideals in $\mathfrak{g}$.

Def. 1.50 A Lie algebra is **solvable**, if $\mathfrak{g}^{(k)} = 0$ for some $k \in \mathbb{N}$.

Ex. $\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{R})$ the subalgebra of upper triangular matrices.

Ex. Abelian Lie algebras.

Def. 1.51 ① A Lie algebra $\mathfrak{g}$ is called **simple**, if $\dim(\mathfrak{g}) > 1$ and $\mathfrak{g}$ and $\{0\}$ are the only ideals of $\mathfrak{g}$.

② $\mathfrak{g}$ is called **semisimple**, if $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ where $\mathfrak{g}_i$ is a simple ideal of $\mathfrak{g}$ $\forall i = 1, \dots, k$.

For semisimple Lie algebras one has $\mathfrak{g} = \underline{[\mathfrak{g}, \mathfrak{g}]}$.

Hence, the intersection of solvable and semisimple Lie algebras is empty.

<u>Thm. 1.52</u> (Levi decomposition)

Suppose $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. Then $\exists$ a semisimple subalgebra $\mathfrak{l} \leq \mathfrak{g}$ s.t.

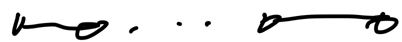
$$\mathfrak{g} = \mathfrak{l} \ltimes \mathfrak{r} \quad \left(\text{as a vector space } \mathfrak{g} = \mathfrak{l} \oplus \mathfrak{r}\right)$$

semidirect product

where $\mathfrak{r}$ is radical of $\mathfrak{g}$, the largest solvable ideal of $\mathfrak{g}$.

<u>Thm. 1.53</u> (Classification of complex simple Lie algebras)

If $\mathfrak{g}$ is a complex simple Lie algebra, then it is isomorphic to either of the following Lie algebras:

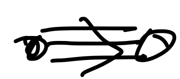


- $\mathfrak{sl}(n,\mathbb{C})$
- $\mathfrak{so}(n,\mathbb{C})$
- $\mathfrak{sp}(n,\mathbb{C})$

} classical families of simple Lie algebras.

- E_6
- E_7
- E_8
- F_4
- G_2

} exceptional ones.



Classifying the real forms of these complex simple Lie algebras gives a classification of the real simple Lie algebras.

1.7 Some more information on representations of Lie groups and algebras

G Lie group

$\varphi : G \times V \longrightarrow V$ $\quad$ V vector space. $\quad$ $(\varphi_g : V \to V$

φ is a complex representation of G, if V is a complex vector space and $\varphi(g, -) : V \to V$ is complex linear $\forall g \in G$.

Def. 1.54 G Lie group and V, W representations of G.

Then a morphism (resp. isomorphism) between V and W is a linear map (resp. a linear isomorphism) $f: V \to W$ s.t. $f(g.v) = g.f(v) \quad \forall g \in G, \forall v \in V$.

Such a morphism is also called a G-equivariant map between the G-represent. V and W.

Some constructions with representations

① Direct sums of representations

V, W representations of G

$\Longrightarrow$ then there is a natural representation of G on $V \oplus W$:

$$G \times V \oplus W \longrightarrow V \oplus W$$
$$(g, (v, w)) \longmapsto (g \cdot v, g \cdot w)$$

② A subrepresentation of a G-representation V is a G-invariant subspace W of V, that is, $g \cdot w \in W$ $\forall g \in G$ $\forall w \in W$.

Restricting the representation of G on V to W gives a representation of G on W.

③ Quotients of representations

Suppose $W \subseteq V$ is a G-invariant subspace, then there is a natural representation of G on V/W:

$$G \times V/W \longrightarrow V/W$$
$$(g, v+W) \longmapsto g \cdot v + W .$$

④ Tensor product of representations: V, W repres. of G

G has a natural representation on $V \otimes W$

$$G \times V \otimes W \longrightarrow V \otimes W$$
$$(g, v \otimes w) \longmapsto (g \cdot v \otimes g \cdot w)$$

⑤ Dual representation

V is a representation of G, then G has a natural representation on V^*:

$$G \times V^* \longrightarrow V^*$$
$$(g, \lambda) \longmapsto g \cdot \lambda : V \to \mathbb{R}$$

$$(g \cdot \lambda)(v) = \lambda(g^{-1} \cdot v).$$

inverse of g.

Def. 1.55 G Lie group, V a represent. of G.

① V is called irreducible, if $\{0\}$ and V are the only G-invariant subspaces of V.

② V is called decomposable, if $\exists$ two G-representations V_1 and V_2 with $\dim(V_1), \dim(V_2) > 0$

s.t. $$V \underset{\substack{\uparrow \\ \text{as}}}{\simeq} \underbrace{V_1 \oplus V_2}_{G\text{-representations}}$$

It is called indecomposable otherwise.

<u>Remark</u> There are representations that are not irreducible but indecomposable!

<u>Def. 1.56</u> A representation V of a Lie group G is called **completely reducible**, if any G-invariant subspace of V has a G-invariant compliment.

If this is the case, then $V \simeq V_1 \oplus \dots \oplus V_k$ for V_i irreducible representations of G.

Lemma 1.57 V, W are representations of G and $f: V \to W$ a morphism between them.

1. $\ker(f) \subseteq V$ and $\operatorname{im}(f) \subseteq W$ are G-invariant subspaces and f induces an isomorphism of representations
$$\bar{f}: V/_{\ker(f)} \xrightarrow{\sim} \operatorname{im}(f).$$

2. If V and W are irreducible, then f is either zero or an isomorphism.

3. If V is a complex irreducible repres. and $W = V$, then f is a complex multiple of the identity on V.

Schur's Lemma

Proof

1 $v \in \ker(f)$ $\qquad f(g \cdot v) = g \cdot \underbrace{f(v)}_{=0} = 0$

and similarly for $\mathrm{Im}(f)$.

2 By irreducibility of V, $\ker(f) = V$ (i.e. $f = 0$)

or $\ker(f) = \{0\}$ by 1.

If $\ker(f) = \{0\}$, then $\mathrm{Im}(f) \neq \{0\}$ and hence $\mathrm{Im}(f) = W$

by 1 and irreducibility of W.

$\Longrightarrow f : V \to W$ is an isomorphism.

③ $f : V \to V$ complex linear map.

$\Longrightarrow$ $\exists$ at least one eigenvalue λ

Denote by W the eigenspace corresp. to λ

Then W is G-invariant: $f(\underline{g\cdot w}) = g\cdot \underbrace{f(w)}_{\lambda w} = \lambda(\underline{g\cdot w})$ $\forall g\in G$ $\forall w \in W$.

By irreducibility of W, ~~[illegible]~~. $V = W$.

$\square$

Def. 1.58 G Lie group

A representation V is called unitary, if $\exists$ a positive definite inner product $\langle\, , \rangle$ on V (Hermitian, if V is complex) s.t.:

$$\langle g.v, g.w\rangle = \langle v, w\rangle \quad \forall g \in G,\ \forall v, w \in V.$$

$$\left(\begin{aligned} \varphi : G \longrightarrow O(V, \langle\, , \rangle) &\subseteq GL(V) \\ U(V, \langle\, , \rangle) &\subseteq GL(V) \end{aligned} \right).$$

Prop. 1.59 Any unitary representation is completely reducible.

Proof. $W \subseteq V$ a G-invariant subspace of a unitary G-repres. V.

Then $W^\perp = \{v \in V : \langle v, w \rangle = 0 \ \forall w \in W\}$ (orthog. compl. w.r.t. to $\langle , \rangle$ of W).

is a G-invariant compliment of W.

Let $v \in W^\perp$, $g \in G$, $w \in W$,

$$\langle g\cdot v, w \rangle = \langle \underbrace{g^{-1}\cdot g\cdot v}_{=v}, g^{-1}\cdot w \rangle = \langle v, \underbrace{g^{-1}\cdot w}_{\in W} \rangle = 0$$

Hence, $W^\perp$ is G-invariant ($V \cong W \oplus W^\perp$).

Thm. 1.60 Let G be a compact Lie group. Then any representation of G is unitary and hence completely reducible.

Proof. see literature, or homework.

Analogously one defines morphisms of representations of Lie algebras, invariant subspaces, direct sums, quotients (also tensor products and dual representations → formula a bit different).

$$\left(\begin{array}{ll} \mathfrak{g} \times V \otimes W \to V \otimes W & X \cdot v \otimes w = Xv \otimes w + v \otimes Xw \\ \mathfrak{g} \times V^* \to V^* & (X \cdot \lambda)(v) := -\lambda(X \cdot v) \quad \forall X \in \mathfrak{g}, \ \forall v \in V, \ \forall w \in W \end{array} \right)$$

and also irreducible, indecomposable, completely reducible.

<u>Unitary</u>: $\varphi : G \to O(V)$, $U(V)$ $\quad \varphi' : \mathfrak{g} \to \mathfrak{o}(V)$, $\mathfrak{u}(V)$.

$\exists$ inner product (Hermitian if V is complex) on V s.t.
$\mathfrak{g}$ acts by $\langle\, , \rangle$ - skew symmetric (resp. skew Hermitian) linear maps: $\langle X.v, w \rangle = -\langle v, Xw \rangle$ $\forall X \in \mathfrak{g}$ and $\forall v, w \in V$.

<u>Prop. 1.61</u> G is a connected Lie group, $\varphi : G \to GL(V)$ a repres. on a finite-dim. vectorspace V and $\varphi' : \mathfrak{g} \to \mathfrak{gl}(V)$ the induced represent. of $\mathfrak{g}$.

(1) A subspace $W \subseteq V$ is G-invariant $\Longleftrightarrow$ it is $\mathfrak{g}$-invariant.

In particular, V is indecomposable, irreducible or completely reducible as a G-repres. $\Longleftrightarrow$ it is as a $\mathfrak{g}$-repres.

(2) V is unitary as a G-repres. $\Longleftrightarrow$ it is unitary as a $\mathfrak{g}$-representation.

<u>Proof.</u> Exercise.

<u>Thm. 1.62.</u> (Thm of Weyl) Suppose $\mathfrak{g}$ is a complex or real finite-dim. semisimple Lie algebra. Then any finite-dim. representation is completely reducible. <u>Proof</u> see literature.