Last week :

$V \xrightarrow{p} M$ equipped with a linear conn. $\nabla : \Gamma(V) \longrightarrow \Gamma(T^*M \otimes V)$

⇝ Parallel transport induced by ∇.

⇝ Horizontal distribution / Horizontal lift :

$$H^{\nabla} \subseteq TV \quad \text{s.t.} \quad TV = \underline{\underline{H^{\nabla}}} \oplus \underline{Ver(V)}$$

$$\begin{array}{c} TV \\ \big\downarrow Tp \\ \underline{TM} \end{array}$$

$$Hor_v : T_xM \longrightarrow H^{\nabla}_v \subseteq T_vV \qquad \xi \in \Gamma(TM)$$

$v \in V_x \qquad Hor_v(\xi_x) =: \xi^{hor}_v$. $\qquad \xi^{hor} : v \longmapsto \xi^{hor}_v$ is a section of H^{∇}.

$$\mathrm{Hor}_v : T_x M \longrightarrow H_v \subseteq T_v V$$

$$\gamma'(0) \longmapsto \frac{d}{dt}\Big|_{t=0} Pt_0^t(\gamma)(v)$$

$\gamma : I \to M$ C^∞-curve, $\gamma(0) = x$.

One can recover ∇ from the horizontal distribution / horizontal lift:

<u>Prop. 3.7</u> Let ∇ be a linear connection on a vector bundle $V \to M$. Then for $\xi \in \Gamma(TM)$, $s \in \Gamma(V)$ one has

$$\nabla_\xi s = \underline{T}s \circ \xi - \xi^{hor} \circ s.$$

$$\left((\nabla_\xi s)(x) \in \mathrm{Ver}_{s(x)}(V) \cong \underline{V_x} \right)$$

<u>Proof</u>. Check this!

In fact one can check that a linear connection ∇ on a vector bundle $V \to M$ is equivalent to the choice of a vector subbundle $H \subset TV$ s.t.

- $TV = H \oplus Ver(V)$
- $\underline{H_{\lambda v} = T_v m_\lambda H_v}$ $\quad \forall v \in V, \forall \lambda \in \mathbb{R}$,

where $m_\lambda : V \to V$ equals $m_\lambda(v) = \lambda v$ scalar multiplication (which is a diffeomorphism for $\lambda \neq 0$).

<u>Prop. 3.8</u> Suppose ∇ is a linear connection on a vector bundle $V \to M$. Then there exist a unique section $R \in \Gamma(\Lambda^2 T^* M \otimes Hom(V, V))$

which is characterized by

$$R(\xi,\eta)(s) = \nabla_\xi \nabla_\eta s - \nabla_\eta \nabla_\xi s - \nabla_{[\xi,\eta]} s \qquad \forall \xi,\eta \in \Gamma(TM)$$
$$\forall s \in \Gamma(V).$$

Proof. The fact that R defines a section of $\Lambda^2 T^*M \otimes \mathrm{Hom}(V,V)$
$\simeq \Lambda^2 T^*M \otimes V^* \otimes V$
is verified straightforwardly $\longrightarrow$ cf. affine connections in Global Analysis.

$\square$

Moreover, given a linear connection ∇ on a vector bundle $V \to M$, we get an induced connection on $V^* \to M$,

$$(\nabla_\xi \lambda)(s) = d(\lambda(s))(\xi) - \lambda(\nabla_\xi s) \qquad \forall \lambda \in \Gamma(V^*)$$
$$\forall s \in \Gamma(V)$$
$$\forall \xi \in \Gamma(TM)$$

$$\Gamma(V) \times \Gamma(V^*) \longrightarrow C^\infty(M, \mathbb{R})$$

$$(s, \lambda) \longmapsto \lambda(s) : x \longmapsto \lambda_x(s_x)$$

If we set $\nabla_\xi f = df(\xi)$ $\quad \forall\, C^\infty(M, \mathbb{R})$

then $(\nabla_\xi \lambda)(s) = \nabla_\xi(\lambda(s)) + \lambda(\nabla_\xi s)$.

One gets linear connection on tensor products of V and V^*:

$$\nabla_\xi(s \otimes s') = \nabla_\xi s \otimes s' + s \otimes \nabla_\xi s'$$

$s \otimes s' \in \Gamma(V \otimes V)$

$\longrightarrow$ connection on $V \otimes V$

$$\nabla_\xi(\lambda \otimes s') = \nabla_\xi \lambda \otimes s' + \lambda \otimes \nabla_\xi s'$$

$\lambda \otimes s' \in \Gamma(V^* \otimes V) = \Gamma(\mathrm{Hom}(V, V))$

The curv. R^* of ∇ on V^*: $R^*(\xi,\eta)(\lambda)(s) =$

$= -\lambda\big(R(\xi,\eta)(s)\big)$

$\forall \xi,\eta \in \Gamma(TM)$

$\forall \lambda \in \Gamma(V^*)$

$\forall s \in \Gamma(V)$.

3.3 Connections on general fiber bundles

Def. 3.9 Suppose $p: E \to M$ is a fiber bundle with standard fiber F. A general connection on $p: E \to M$ is a vector subbundle $H \subseteq TE$ s.t. $TE = H \oplus Ver(E)$. $(*)$

Denote by $\Pi: TE \to Ver(E)$ and $\chi = Id_{TE} - \Pi : TE \to H$ the natural projections corresponding to $(*)$.

(General connection is equiv. to a projection $\Pi : TE \to Ver(E)$ $(\Pi|_{Ver(E)} = Id_{Ver(E)})$ with kernel H.

- General connection H induces a horizontal lift map:

$$\mathrm{Hor} : \Gamma(TM) \longrightarrow \Gamma(H) \subseteq \Gamma(TE)$$
$$\xi \longmapsto \xi^{hor}$$

which is linear of smooth fcts.
ξ^{hor} is the unique vector field on E s.t. $T_u p\, \xi^{hor}_u = \xi(p(u))$ $\forall u \in E$
and $\xi^{hor}(u) \in H_u$ $\forall u \in E$.

Conversely, any horizontal lift of vector fields with this property comes from a general connection.

- General connection can be viewed as a notion of parallel transport
$\gamma : I \to M \rightsquigarrow Pt_{t_1}^{t_2}(\gamma) : E_{\gamma(t_1)} \longrightarrow E_{\gamma(t_2)}$ diffeomorphism

- General connection can be seen as a general covariant derivative:

$$\nabla_{\varsigma} s = \underline{\underline{Ts \circ \varsigma}} - \varsigma^{hor} \circ s \in \Gamma(Ver(E) \to M).$$

$\forall s \in \Gamma(E), \forall \varsigma \in \Gamma(TM)$.

<u>Def. 3.10</u> The curvature of a general connection on a fiber bundle $p: E \to M$ is defined as follows.

$$R(\varsigma, \eta) = -\Pi(\underline{[\chi(\varsigma), \chi(\eta)]}) \qquad \forall \varsigma, \eta \in \Gamma(TE).$$

- R is horizontal, i.e. it vanishes upon insertion of any section of $Ver(E) \to E$.
- $\Pi \circ \chi = 0 \implies R$ is bilinear over $C^\infty(E, \mathbb{R})$.

Therefore $R \in \Gamma(\Lambda^2 T^*E \otimes Ver(E)) =: \Omega^2(E, Ver(E))$.

In fact $R \in \Omega^2_{hor}(E, Ver(E)) = \{\alpha \in \Omega^2(E, Ver(E)) :$
$\alpha(\xi, -) = 0 \ \forall \xi \in \Gamma(Ver(E))\}$

By the Frobenius Thm., $R = 0 \iff H$ is integrable.

3.4 Principal connections on principal bundles

<u>Def. 3.11</u> Suppose $p: P \to M$ is a principal G-bundle for a Lie group G. Then a principal connection is a general connection on $p: P \to M$, $H \subseteq TP$ s.t. $T_u r^g H_u = H_{u \cdot g}$ $\forall u \in P$, $\forall g \in G$, $(*)$

where $r^g: P \to P$ denotes the principal right-action by $g \in G$ on P.

Denote by $\Pi: TP \to Ver(P) =: V(P)$ the corresp. vertical projection and by $\mathcal{K} = Id_{TP} - \Pi: TP \to H$ the corresp. horizontal projection.

$(*)$ is equivalent to $\Pi(u\cdot g) \circ T_u r^g = T_u r^g \circ \Pi(u)$.

$p \circ r^g = p \implies T_u r^g \, V_u(P) = V_{u\cdot g}(P)$.

<u>Lemma 3.12</u> $p: P \to M$ principal G-bundle with principal connection $H \subseteq TP$.

① For any $\xi \in \Gamma(TM)$, the horizontal lift ξ^{hor} (w.r.t. H) satisfies $(r^g)^* \xi^{hor} = \xi^{hor}$ $\forall g \in G$.

② The parallel transport w.r.t. H satisfies: For $\gamma: I \to M$

$$Pt(\gamma)(u \cdot g) = (Pt(\gamma)(u)) \cdot g \qquad u \in P_{\gamma(t_1)}.$$

i.e. $Pt_{t_1}^{t_2}(\gamma): P_{\gamma(t_1)} \to P_{\gamma(t_2)}$ is G-equiv.

Proof.

(1) Fix $x \in M$, $u \in P_x$ and let $\xi \in \Gamma(TM)$.

$$\left((r_g)^*\xi^{hor}\right)(u) = (T_u r_g)^{-1} \underbrace{\xi^{hor}(u \cdot g)}_{\in H_{ug}} \in H_u$$

and
$$\underbrace{T_u p\, (T_{ug}}_{T_{ug}(p \circ r_{g^{-1}}) = T_{ug}p} r_{g^{-1}}\, \xi^{hor}(u \cdot g)) = T_{ug}\xi^{hor}(u \cdot g) = \xi(p(u \cdot g)) = \xi_x$$

$\Longrightarrow$ $\xi^{hor}(u) = \left((r_g)^*\xi^{hor}\right)(u)$ by uniqueness.

② Follows also easily along the same pattern.

<u>Def. 3.13</u> $p: P \to M$ principal G-bundle. For any $X \in \mathfrak{g}$ and $u \in P$ set

$$\tilde{X}(u) := \frac{d}{dt}\Big|_{t=0} u \cdot \exp(tX) = \frac{d}{dt}\Big|_{t=0} r^{\exp(tX)}(u) \in T_u P$$

Then $\tilde{X} \in \Gamma(TP)$ is a vector field on P, called **the fundamental vector field generated by $X \in \mathfrak{g}$**.

(In principal bundle chart $\tilde{X}$ corresp. to left-inv. vector field L_X on G, which implies $\tilde{X}$ is smooth).

<u>Lemma 3.14</u> The vertical bundle of a principal G-bundle $p: P \to M$ is trivialized by the fundamental vector fields, i.e.

$$\begin{tikzcd} P \times \mathfrak{g} \arrow[rr, "\sim"] \arrow[dr, "pr_1"'] & & V(P) \arrow[dl] \\ & P & \end{tikzcd} \qquad (u, X) \mapsto \zeta_X(u) \quad (*)$$

is an isomorphism of vector bundles.

<u>Proof</u> $p(u \cdot \exp(tX)) = p(u) \underset{\text{differentiating}}{\rightsquigarrow} T_u p\, \zeta_X(u) = 0$, i.e. $\zeta_X(u) \in V_u(P) \subseteq T_u P$.

and $(*)$ is evidently injective $(\zeta_X(u) = 0$

$\left(u\cdot \exp(tX) = u \ \ \forall \iff \exp(tX) = e \ \ \forall t \iff X = 0 . \right)$.

and hence $\sharp$ is an isomorphism by dimensional reasons.

$\left(G \longrightarrow P_{p(u)}, \ \ g \longmapsto u\cdot g \ \text{ is a diffeo. diff. } \ \mathfrak{g} \xrightarrow{\sim} T_u P_{p(u)} = V_u P \right)$.

<u>Notation 3.15</u>: $p : P \to M$ principal G-bundle, $\mathbb{V}$ G-represent. ($\rho : G \to GL(\mathbb{V})$).

- We write $\Omega^k_{hor}(P, \mathbb{V}) := \{ \alpha ; \Gamma(TP) \times \dots \times \Gamma(TP) \longrightarrow \mathbb{V} : C^\infty(M, \mathbb{R})\text{-linear}$ in each entry and $i_s \alpha = 0 \ \forall s \in \Gamma(VP) \}$ for the space of $\mathbb{V}$-valued horizontal k-forms.
- $\alpha \in \Omega^k(P, \mathbb{V})$ is called **G-equivariant**, if $(r^g)^* \alpha = \rho(g^{-1}) \circ \alpha$ $\forall g \in G$.

We write $\Omega^k(P, \mathbb{V})^G = \{ \alpha \in \Omega^k(P, \mathbb{V}) : \alpha \text{ is } G\text{-equiv.} \}$.

Prop. 3.16 There is a bijection

$$\Omega^k_{hor}(P, \mathbb{V})^G \xrightarrow{\sim} \Gamma\left(\Lambda^k T^*M \otimes (P \times_G \mathbb{V}\right) = \Omega^k(M, P \times_G \mathbb{V}).$$

Proof. Recall $V := \underline{P \times_G \mathbb{V}}$, $\tilde{\alpha} \in \Omega^k(M, V)$.

For $u \in P_x$ and tangent vectors $\xi_1, \dots, \xi_k \in T_u P$, there is unique element $\alpha(u)(\xi_1, \dots, \xi_k) \in \mathbb{V}$ s.t.

$$(*) \quad \tilde{\alpha}(p(u))(\underline{T_u p}\, \xi_1, \dots, \underline{T_u p}\, \xi_k) = [u, \alpha(u)(\xi_1, \dots, \xi_k)].$$

$\alpha(u) : T_uP \times \dots \times T_uP \longrightarrow \mathbb{V}$ defines a k-linear, alternating map, which vanishes $\Longleftrightarrow$ one vector ξ_i is a vertical vector.

It is easy to verify that α depends smoothly on u.

Hence, $\alpha \in \Omega^k_{hor}(P, \mathbb{V})$. For $g \in G$, $p \circ r^g = p$ implies $Tp \circ Tr^g \xi_i = Tp\, \xi_i$ and thus

$$[u, \alpha(u)(\xi_1, \dots, \xi_k)] = [u \cdot g, \alpha(u \cdot g)(Tr^g \xi_1, \dots, Tr^g \xi_k)]$$

and so $\alpha(u \cdot g)(Tr^g \xi_1, \dots, Tr^g \xi_k) = \rho(g^{-1}) \circ \alpha(u)$.

Conversely, given $\alpha \in \Omega^k_{hor}(P, \mathbb{V})^G$, one can use $(*)$ to define a $\underline{\alpha} \in \Omega^k(M, \mathcal{V})$. $\square$

<u>Yesterday</u>: • $P \to M$ principal G-bundle, G-repres. $\mathbb{V}$, $V := P \times_G \mathbb{V}$

$\mathrm{Ver}(P) \to P$ is trivialized by fundamental vector fields.

• $\underline{\Omega^k_{hor}(P, \mathbb{V})^G} \xrightarrow{\sim} \Gamma(\Lambda^k T^*M \otimes V) = \underline{\Omega^k(M, V)}$

<u>Thm. 3.17</u> $p : P \to M$ principal G-bundle

① Any principal connection $H \subseteq TP$ can be equivalently encoded by a connection form, that is, a G-equivariant 1-form on P with values in $\mathfrak{g}$ $\underline{\gamma} \in \Omega^1(P, \mathfrak{g})^G$ s.t. $\gamma(e_X) = X \quad \forall X \in \mathfrak{g}$.

One has $H = \ker(\gamma : TP \to \mathfrak{g})$.

② Any principal bundles admits a principal connection and the space of all principal connections is an affine space over the vector space $\Gamma(T^*M \otimes P \times_G \mathfrak{g}) \cong \Omega^1_{hor}(P, \mathfrak{g})^G$.

Proof.

① A principal connection $H \subseteq TP$ is equivalent to a vertical projection $\Pi : TP \to V(P)$ $\left(\Pi|_{V(P)} = \mathrm{Id}_{V(P)}\right)$ that is compatible with Tr_g $\forall g \in G$.

$\underbrace{\Pi(u)(\xi)}_{\in V_u P} = \zeta_{\gamma(\xi)}(u)$ for a unique $\gamma(\xi) \in \mathfrak{g}$

$, \xi_u \in T_u P$ by Lemma 3.14.

$\gamma : TP \to \mathfrak{g}$

- $\gamma(u)(\ell_X(u)) = X$, since $\Pi|_{V_uP} = \mathrm{Id}_{V_uP}$.

- G-compatibility of Π is equiv. to G-equivariancy of γ :

For any $X \in \mathfrak{g}$, $g \in G$, $\quad g^{-1}\exp(tX)\, g = \exp(t\,\mathrm{Ad}(g^{-1})(X))$.

$$\Longrightarrow \quad \ell_{\mathrm{Ad}(g^{-1})(X)}(u\cdot g) = \frac{d}{dt}\Big|_{t=0} u\cdot g\cdot \exp(t\,\mathrm{Ad}(g^{-1})(X)) = \frac{d}{dt}\Big|_{t=0} \overbrace{u\cdot \exp(tX)\cdot g}^{r^g(u\cdot \exp(tX))}$$

$$= T_u r^g\, \ell_X(u) \qquad (r^g)^*\gamma$$

$$\Pi(u\cdot g)(T_u r^g\, \ell_X(u)) = \ell_{\gamma(u\cdot g)(T_u r^g \ell_X(u))}(u\cdot g)$$

$$\|$$

$$T_u r^g \circ \Pi(u)(\ell_X(u)) = T_u r^g\, \ell_X(u) = \ell^{(u\cdot g)}_{\mathrm{Ad}(g^{-1})(X)}$$

$$\gamma(u\cdot g)(T_u r^g \ell_X(u)) \;\|\; \mathrm{Ad}(g^{-1})\circ\gamma(u)(\ell_X(u))$$

(2) Existence: locally clear + partitions of unity (cf. linear connection).

Freedom: Suppose γ and $\tilde{\gamma}$ are two principal connections,

$\gamma - \tilde{\gamma} \in \Omega^1_{hor}(P, \mathfrak{g})^G$, since γ and $\tilde{\gamma}$ reproduce the generators of fund. vector fields.

$\cong$

$\underline{\Gamma(T^*M \otimes P \times_G \mathfrak{g})}$ and Lemma 3.14.

Moreover, for any $\alpha \in \Omega^1_{hor}(P, \mathfrak{g})^G$,

$$\gamma + \alpha \in \Omega^1(P, \mathfrak{g})^G$$

and $(\gamma + \alpha)(e_X) = \underbrace{\gamma(e_X)}_{= X} + \underbrace{\alpha(e_X)}_{=0} = X$.

$\square$

Prop. 3.18 $p: P \to M$ a principal G-bundle equipped with a principal connection $\gamma \in \Omega^1(P, \mathfrak{g})$. Then the curvature

$$R \in \Omega^2_{hor}(P, \underline{\underline{VP}})$$

can be identified with a $\mathfrak{g}$-valued two form of the form

$$\rho \in \Omega^2_{hor}(P, \underline{\mathfrak{g}})^G \simeq \Gamma(\Lambda^2 T^*M \otimes P \times_G \mathfrak{g}).$$

Moreover, $\rho(\xi, \eta) := d_\gamma(\xi, \eta) + [\gamma(\xi), \gamma(\eta)] \quad \forall \xi, \eta \in \Gamma(TP)$.

d_γ is defined as for $\Omega^1(P, \mathbb{R}) = \Omega^1(P)$.

Proof. $\Pi \longleftrightarrow \gamma \qquad \mathcal{X} = \mathrm{Id} - e_{\gamma(-)} \qquad \mathcal{X}(\xi) = \xi - e_{\gamma(\xi)}$

$$p(\xi,\eta) = -\gamma\Big(\big[\underbrace{\xi - e_{\gamma(\xi)}}_{\in H\text{-}\ker\gamma},\ \underbrace{\eta - e_{\gamma(\eta)}}_{\in \ker(\gamma) = H}\big]\Big) = d\gamma\big(\xi - e_{\gamma(\xi)},\ \eta - e_{\gamma(\eta)}\big)$$

$$= d\gamma(\xi,\eta) - \underbrace{d\gamma(e_{\gamma(\xi)},\eta)}_{-[\gamma(\xi),\gamma(\eta)]} - \underbrace{d\gamma(\xi, e_{\gamma(\eta)})}_{-[\gamma(\xi),\gamma(\eta)]} + \underbrace{d\gamma(e_{\gamma(\xi)}, e_{\gamma(\eta)})}_{-[\gamma(\xi),\gamma(\eta)]}$$

e_x has flow $r^{\exp(tx)}$

$$r^{\exp(tx)*}\gamma = \mathrm{Ad}(\exp(-tx))\circ\gamma \qquad (G\text{-equiv. of } \gamma).$$

Differentiating at $t=0$: $\mathcal{L}_{e_x}\gamma = -\mathrm{ad}(x)\circ\gamma$

$$\| \quad i_{e_x}\circ d\gamma + d(\underbrace{i_{e_x}\gamma}_{=0})$$

$$\Longrightarrow (\mathcal{L}_{e_x}\gamma)(\eta) = d\gamma(e_x, \eta) = -[x, \gamma(\eta)] \quad \forall \eta \in \Gamma(TP).$$

$$(r^{g*}\rho) = (\underbrace{(r^g)^* d}\gamma) + r^{g*}([\gamma(-), \gamma(-)])$$

$$r^{g*}([\gamma(-), \gamma(-)]) = [(r^{g*}\gamma)(-), (r^{g*}\gamma)(-)] = [Ad(g^{-1})\circ\gamma(-), Ad(g^{-1})\circ\gamma(-)] = Ad(g^{-1})\circ[\gamma(-), \gamma(-)].$$

$$(r^g)^* d\gamma = d\, r^{g*}\gamma = d(\underline{Ad(g^{-1})}\circ\gamma) = Ad(g^{-1})\circ d\gamma$$

$$= \underline{Ad(g^{-1})\circ\rho}$$

$\forall g \in G$.

3.5 Induced connections on associated bundles.

Suppose $p: P \longrightarrow M$ is a principal G-bundle. Then any principal connection γ induces (linear) connections on all associated (vector) bundles.

Let F be a mfld. equipped with a G-action $G \times F \to F$.

$\pi: P \times_G F \longrightarrow M$ associated bundle

$$\begin{tikzcd} P \times F \arrow[r, "q"] \arrow[d, "pr_1"'] & P \times_G F \arrow[d, "\pi"] \\ P \arrow[r, "p"] & M \end{tikzcd} \qquad \pi \circ q = p \circ pr_1$$

- TG is again a Lie group with multipl. $T\mu : TG \times TG \to TG$ (neutral element $(e, 0) \in T_e G$).

- TP is a principal TG-bundle with principal right action $Tr^g : TP \to TP$

- $Tq : TP \times TF \to T(P \times_G F)$ induces an identification / isomorph.

$$\underline{TP \times_{\underline{TG}} TF} \simeq \underline{T(P \times_G F)},$$

$P \hookrightarrow TP$, $G \hookrightarrow TG$ via the zero section.

$Tq|_{P \times TF} : \underline{P \times TF} \to T(P \times_G F)$ induces an identification of $\underline{P \times_G TF \simeq Ver(P \times_G F)}$.

Given a principal connection γ on P with corresp. horizontal distribution H.

$$T_q : TP \times TF \to T(P \times_G F).$$

For $(u,f) \in P \times F$

$$T_{(u,f)}q\big|_{H_u \times \{0_f\}} : H_u \times \{0_f\} \longrightarrow T_{[u,f]}(P \times_G F)$$

is an injection. We write $H_{[u,f]} := T_{(u,q)}q(H_u \times \{0_f\})$.

$$q \circ r^g \times \ell_{g^{-1}} = q$$

$$Tq \circ Tr^g \times T\ell_{g^{-1}} = Tq$$

This is well-defined:

$$T_{(u \cdot g, g^{-1}f)}q(H_{u \cdot g} \times \{0_{g^{-1}f}\}) =$$

$$= T_{(ug,\, g^{-1}f)}\, g \left(\underline{T_u r_g H_u} \times \{0_{g^{-1}f}\} \right)$$

$$= T_{(u,f)}\, g \left(H_u \times \{0_f\} \right).$$

$$\mathcal{H}_{[u,f]} = \{ [s(u), 0_f] \in TP \times_{TG} TF \ : \ s(u) \in H_u,\ u \in P \}$$

Hence, $\mathcal{H}_{[u,f]} \subseteq T_{[u,f]}(P \times_G F)$ is a general connection on $P \times_G F$.

Thm. 3.19 Suppose $p : P \to M$ is a principal G-bundle, $\phi : G \to GL(\mathbb{V})$ a repres. of G and set $V := P \times_G \mathbb{V} \to M$.

Let $\gamma \in \Omega^1(P, \mathfrak{g})$ be a principal connection on P with horiz. distr. H^γ and let H^V be the induced connection on $V \to M$.

(1) $H^V \subseteq TV$ is a linear connection on V with horizontal lift given by

$$\begin{aligned} \xi &\longmapsto \Phi_q\left(\left(\xi^{hor}, 0\right)\right) \\ \Gamma(TM) &\longrightarrow \Gamma(V) \end{aligned}$$

where $\xi^{hor} \in \Gamma(TP)$ is the horizontal lift w.r.t. γ.

(2) Let $s \in \Gamma(V)$ with corresp. fct. $f_s \in C^\infty(P, \mathbb{V})^G$ and $\xi \in \Gamma(TM)$, then the covariant deriv. $\nabla_\xi s \in \Gamma(V)$ (induced by H^V) corresponds to the fcts. $\xi^{hor} \cdot f_s \in C^\infty(P, \mathbb{V})^G$.

(3) Parallel transport on V w.r. to H^V (or ∇) along a curve c on M is given by

$$(Pt^\nabla)_0^t (c)([u,v]) = [Pt_0^t(c)(u), v] ,$$

where $Pt_0^t(c)$ is the parallel transport of c w.r. to $\mathfrak{J}$.

(4) For $\xi, \eta \in \Gamma(TM)$, $s \in \Gamma(V)$, $R(\xi,\eta)(s) = \nabla_\xi \nabla_\eta s - \nabla_\eta \nabla_\xi s - \nabla_{[\xi,\eta]} s \in \Gamma(V)$

corresp. to the G-equiv. fct. $\phi'(\rho(\xi^{hor}, \eta^{hor})) \circ f_s : P \to \mathbb{V}$,

where ρ is the curvature of γ.

<u>Proof.</u>

① Statement about horizontal lifts is clear.

Also, $H^{\mathbb{V}}$ is a linear connection:

$m_\lambda = P[\tilde{m}_\lambda] : P \times_G \mathbb{V} \to P \times_G \mathbb{V}$ with $\tilde{m}_\lambda : \mathbb{V} \to \mathbb{V}$ linear map.
$v \mapsto \lambda v$

is induced by $q \circ \mathrm{Id}_P \times \tilde{m}_\lambda$.

$$T_{(u,v)}(q \circ \mathrm{Id}_P \times \tilde{m}_\lambda) = T_{(u,\lambda v)} q \circ \underbrace{T_u \mathrm{Id}_P}_{\mathrm{Id}_{T_u P}} \times T_v \tilde{m}_\lambda = T_{(u,\lambda v)} q \circ \mathrm{Id}_{T_u P} \times \tilde{m}_\lambda$$

implies $H^V_{m_\lambda([u,v])} = H^V_{[u,\lambda v]} = T_{[u,v]} m_\lambda H^V_{[u,v]}$.

(2) Follows from $\nabla_\xi s = Ts \circ \xi - \xi^{hor} \circ s$.

(3) clear ✓

(4) By (2) $\nabla_\xi \nabla_\eta s - \nabla_\eta \nabla_\xi s - \nabla_{[\xi,\eta]} s$ corresp. to the function

$$\underbrace{\left([\xi^{hor}, \eta^{hor}] - [\xi,\eta]^{hor} \right)}_{= \rho(\xi^{hor}, \eta^{hor})} \cdot f_s .$$

(5) Suppose $f \in C^\infty(P, V)^G$.

$$(\ell_X \cdot f)(u) = \frac{d}{dt}\Big|_{t=0} f(\underbrace{u \cdot \exp(tX)}) = \cancel{\text{zero}} - \phi'(X) \cdot f(u) \quad \square$$

$\underline{\exp(tX)^{-1}} \, \underline{f(u)}$

$\mathcal{F}(M) \to M$ frame bundle

<u>principal connection</u> $\rightsquigarrow$ lin linear connections on all tensor bundles

Any linear connection ∇ on a vector bundle $V \to M$ is induced from a principal connection on its frame bundle $Fr(V) \to M$. Conversely, any principal connection on $Fr(V) \to M$ induces a linear connection on $V := Fr(V) \times_{GL(\mathbb{V})} \mathbb{V}$.

Indeed, given ∇ on $V \to M$, let $\hat{\gamma} \in \Omega^1(Fr(V), \underline{gl(\mathbb{V})})$ be any principal connection on $Fr(V) \to M$. Then we get an induced linear connection $\hat{\nabla}$ on $V = Fr(V) \times_{GL(\mathbb{V})} \mathbb{V}$.

Then $A(\xi, s) := \hat{\nabla}_\xi s - \nabla_\xi s$. $\quad (A \in \Gamma(T^*M \otimes V^* \otimes V))$

$$V^* \otimes V \simeq Fr(V) \times_{GL(\mathbb{V})} \mathbb{V}^* \otimes \mathbb{V} = Fr(V) \times_{GL(\mathbb{V})} gl(\mathbb{V}).$$

$$\to A \in \Gamma(T^*M \otimes Fr(V) \times_{GL(\mathbb{V})} g(\mathbb{V})) \simeq \underline{\Omega^1_{hor}(Fr(V), gl(\mathbb{V}))^{GL(\mathbb{V})}}$$

We can form $\gamma := \hat{\gamma} + A$, which is again a principal connection on $Fr(V) \to M$ by Thm. 3.17, which by

Construction induces ∇ on $V \to M$.

Reductions $\underline{P} \hookrightarrow Fr(M)$ of structure group of the frame bundle of M can be interpreted as geometric structures on M.

$(TM = P \times_G \mathbb{R}^n$

Affine connections induced from principal connections of P are then affine connections compatible with the geometric structure (given by the reduction).

<u>Example</u>: $O(n)$ - reduction ~~for~~ $\rightleftharpoons$ metrics g on M.

Affine connections induced from principal $O(n)$ - connections on the orthonormal frame bundle corresp. to affine conn.

∇ s.t. $\nabla g = 0$ $\quad \left(\xi \cdot g(s,\eta) = g(\nabla_\xi s, \eta) + g(s, \nabla_\xi \eta) \right)$

$(M, [g])$ conformal mf.

Exams: 31.5, 4.6, 11.6