TUTORIAL 1

1. Suppose V is a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

$$GL(V) = \{\text{linear isomorphism } V \to V\}$$

Its Lie algebra is $\mathfrak{gl}(V) = \{\text{linear maps } V \to V\}$
$$= V^* \otimes V$$
with Lie bracket the commutator of endomorphism of V:
$$[X,Y] = X \circ Y - Y \circ X \quad \forall X, Y \in \mathfrak{gl}(V).$$

Via choice of basis for V: $\quad GL(V) \simeq GL(n, \mathbb{K})$
$$\mathfrak{gl}(V) \simeq \mathfrak{gl}(n, \mathbb{K})$$

$[\ ,\ ]$ = commutator of matrices.

Many Lie groups arise as Lie subgroups of $GL(V)$ consisting of linear isomorphism of V preserving some additional geometric structure on V.

- Suppose V is a real vector space.

Fix an orientation on V.

$$GL_+(V) = \{ A \in GL(V) : A \text{ is orientation preserving} \}.$$

Via choice of basis of V: $GL_+(V) = GL_+(n, \mathbb{R}) =$

$= \{ A \in GL(n,\mathbb{R}) : A \text{ preserves standard orientation on } \mathbb{R}^n \}$

$$= \{ A \in GL(n,\mathbb{R}) \; , \; \det(A) > 0 \} .$$

$GL(V) \cong GL(n,\mathbb{R})$ has two connected components and $GL_+(V) = (GL(V))_0$.

If we fix not only an orientation but a volume form $\omega \in \Lambda^n V^*$ on V,

$$SL(V) = \{ A \in GL(V) : A \text{ preserves } \omega \} \subseteq GL_+(V)$$

$$SL(V) \simeq SL(n,\mathbb{R}) = \{ A \in GL(n,\mathbb{R}) : A \text{ preserves standard volume form on } \mathbb{R}^n \}$$

$$\left(\lambda_1 \wedge \dots \wedge \lambda_n \text{, where } \{\lambda_i\} \text{ is dual basis to the standard basis} \right)$$

$$= \{ A \in GL(n,\mathbb{R}) : \det(A) = 1 \}.$$

- subgroup: $\det(AB) = \det(A)\det(B)$.
- $SL(n,\mathbb{R}) = f^{-1}(0)$ $\quad f = \det - 1 : GL(n,\mathbb{R}) \to \mathbb{R}$

  C^∞ and regular.

$$T_{Id} f = T_{Id} \det = \mathrm{tr} : \mathfrak{gl}(n,\mathbb{R}) \to \mathbb{R}$$

$\Longrightarrow SL(n,\mathbb{R}) \subseteq GL(n,\mathbb{R}) \subseteq M_n(\mathbb{R})$ is a submfd.

$$\Longrightarrow \quad \mathfrak{sl}(n,\mathbb{R}) = \ker(T_{Id} f) = \{X \in \mathfrak{gl}(n,\mathbb{R}) : \mathrm{tr}(X) = 0\}.$$

$$\dim(SL(n,\mathbb{R})) = \dim(\mathfrak{sl}(n,\mathbb{R})) = n^2 - 1.$$

For a complex vector space V fix a complex volume form ω giving rise to the standard orientation on V.

$$GL(V) \cong GL(n,\mathbb{C}) \subseteq \mathbb{C}^{n^2} \text{ open subset.}$$

$GL(n,\mathbb{C})$ is connected : $\det_{\mathbb{C}} : GL(n,\mathbb{C}) \longrightarrow \mathbb{C}\setminus\{0\}$

connected.

$$SL(V) = \{A \in GL(V) : A \text{ preserves } \omega\}$$
$$\cong \{A \in GL(n,\mathbb{C}) : \det_{\mathbb{C}}(A) = 1\}.$$

It is a complex Lie subgroup, since $f = \det_{\mathbb{C}} - 1$ is holomorphic.

$$\mathfrak{sl}(n,\mathbb{C}) = T_{\mathrm{id}} SL(n,\mathbb{C})$$

$$= \{X \in \mathfrak{gl}(n,\mathbb{C}) : \mathrm{tr}(X) = 0\}.$$

$$\dim_{\mathbb{C}}(SL(n,\mathbb{C})) = n^2 - 1$$

$$\dim_{\mathbb{R}}(SL(n,\mathbb{C})) = 2n^2 - 2.$$

- V is a real vector space. Equipp it with a non-degenerate symmetric bilinear form $b: V \times V \to \mathbb{R}$ of signature $\underline{(p,q)}$ ($\dim(V) = p + q =: n$).

$$O(V,b) = \{ A \in GL(V) : \underline{b(Av, Aw) = b(v,w)} \ \forall v, w \in V \}.$$

Up to isomorphism, $\exists$ a unique such bilinear form of sign. (p,q) on V, hence we can find an identification

$$O(V,b) = O(\mathbb{R}^n, \langle , \rangle) =: \underline{O(p,q)}$$

where $\langle x, y \rangle := x^t \underbrace{\begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}}_{I_{p,q}} y$ is the standard inner product of $\mathbb{R}^n$ of signature (p,q).

$$O(p,q) = \{ A \in GL(n,\mathbb{R}) : A^t I_{p,q} A = I_{p,q} \}.$$

$$\parallel$$

$$f^{-1}(0) \qquad f(A) = A^t I_{p,q} A - I_{p,q} \quad \text{cts-map}.$$

$\underline{f : GL(n, \mathbb{R}) \longrightarrow M_n^{sym}(\mathbb{R})} \longleftarrow$

$$T_A f(X) = \frac{d}{dt}\Big|_{t=0} f(A + tX) = \underline{X^t I_{p,q} A + A^t I_{p,q} X}.$$

$S \in M_n^{sym}(\mathbb{R}) \qquad X = \frac{1}{2} A I_{p,q} S \qquad T_A f(X) = S.$

$\Longrightarrow O(p,q) \subseteq M_n(\mathbb{R})$ is a submfd.

$\mathfrak{o}(p,q) := \ker(T_{\mathrm{id}} f) = \{ X \in \mathfrak{gl}(n, \mathbb{R}) : \underline{X^t I_{p,q}} + I_{p,q} X = 0 \}$

$q = 0 \quad : \quad \mathfrak{o}(n) := \mathfrak{o}(n, 0) = \{ X \in \mathfrak{gl}(n, \mathbb{R}) : \underline{X^t = -X} \}$

$\dim(O(p,q)) = \frac{n(n-1)}{2}.$

$$A^t I_{p,q} A = I_{p,q} \Longrightarrow \det(A^t)\det(I_{p,q})\det(A) = \det(I_{p,q})$$

$$\Longleftrightarrow \det(A)^2 = 1$$

$$\Longleftrightarrow \underline{\det(A) = \pm 1}.$$

$$SO(p,q) = O(p,q) \cap SL(n,\mathbb{R}) = O(p,q) \cap GL_+(n,\mathbb{R}).$$

$$\begin{pmatrix} I_p & \\ & -I_q \end{pmatrix} \quad \mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$$

$$\underline{O_0(p,q)} \subset SO(p,q) \subseteq O(p,q) \Longrightarrow \underline{so}(p,q) = \underline{o}(p,q)$$

$$so(n) = o(n)$$

$\nearrow$ has two connected components

For $p = 0$: $O_0(n) = SO(n)$

: V is a complex vector space, then $\exists$ up to isomorphism only one non-degenerate symmetric bilinear form

$$b : V \times V \to \mathbb{C} \quad \text{on } V.$$

(no signature over $\mathbb{C}$!).

$$(V, b) \simeq (\mathbb{C}^n, \langle , \rangle)$$

$\uparrow$ standard inner product $\langle x, y \rangle = x^t y$

$$O(n, \mathbb{C}) = \{ A \in GL(n, \mathbb{C}) : A^t = A^{-1} \}$$

$$\mathfrak{o}(n, \mathbb{C}) = \{ X \in \mathfrak{gl}(n, \mathbb{C}) : X^t = -X \}$$

$\mathfrak{so}(n, \mathbb{C})$

$$\underline{SO(n, \mathbb{C})} = \underline{O_0(n, \mathbb{C})}$$

complex Lie groups of complex dim. $\frac{n(n-1)}{2}$

. V is complex vector space ..

and let $b : V \times V \to \mathbb{C}$ be a Hermitian inner product of signature (p,q)

(. b is non-degenerate $\lambda \in \mathbb{C}$
. $b(\lambda v, w) = \bar{\lambda} b(v,w)$
$b(v, \lambda w) = \lambda b(v,w)$
$b(v,w) = \overline{b(w,v)}$.

$U(V,b) = \{ A \in GL(V) : b(Av, Aw) = b(v,w) \}$

$\exists$ up to isomorphism only one such b ,

hence we can find an identification

$U(V,b) =: U(p,q) = \{ A \in GL(n, \mathbb{C}) : \langle Av, Aw \rangle = \langle v, w \rangle \ \forall v, w \in \mathbb{C}^n \}$

$$\cong \{ A \in GL(n,\mathbb{C}) : \underline{\overline{A}^t I_{p,q} A = I_{p,q}} \}$$

where $\langle v, w \rangle = \overline{v}^t I_{p,q} w$ is the standard ~~inner~~ Hermitian inner product of signature (p,q).

$$U(n) := U(n,0) = \{ A \in GL(n,\mathbb{C}) : \overline{A}^t = A^{-1} \}$$

$n=1$ $\quad U(1) = \{ a \in \mathbb{C}\setminus\{0\} : \overset{|a|^2}{\overset{\parallel}{\overline{a} a}} = 1 \} = \{ a \in \mathbb{C}\setminus\{0\} : |a| = 1 \}$

$= S^1$

$=$ circle group.

$U(p,q) = f^{-1}(0)$ $\qquad f : GL(n,\mathbb{C}) \to \underline{W} = \{ X \in gl(n,\mathbb{C}) : \bar{X}^t = X \}$

$f(A) = \bar{A}^t I_{p,q} A - I_{p,q}$

smooth and regular.

$$T_A f(X) = \frac{d}{dt}\Big|_{t=0} f(A+tX) = \bar{A}^t I_{p,q} X + \bar{X}^t I_{p,q} A.$$

$\Rightarrow U(p,q) \subseteq \mathbb{C}^{n^2}$ is a real submfd.

and $u(p,q) = T_{id} U(p,q) = \{ X \in gl(n,\mathbb{C}) : I_{p,q} X + \bar{X}^t I_{p,q} = 0 \}$

$u(n) = \{ X \in gl(n,\mathbb{C}) : \underline{X = -\bar{X}^t} \}.$

$\dim(U(p,q)) = n^2 = \underline{\underline{n(n-1)}} + n.$ $\qquad \cancel{X \in u(n)\ u \in \mathbb{C}}$

Note $\mathfrak{u}(n)$ is not a complex vector space:

$$X \in \mathfrak{u}(n),\ a \in \mathbb{C} \ : \ \overline{aX}^t = \bar{a}\,\overline{X}^t = -\bar{a}X \neq -aX$$

unless $a \in \mathbb{R}$.

($U(n)$ is only a real Lie subgroup of $GL(n,\mathbb{C})$!).

$U(2)$ has Lie algebra

$$\mathfrak{u}(2) = \{ X \in \mathfrak{gl}(2,\mathbb{C}) : X = -\overline{X}^t \}$$

$$= \left\{ \begin{pmatrix} ix & a+ib \\ -a+ib & iy \end{pmatrix} : x, y, a, b \in \mathbb{R} \right\}.$$

Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\{ i\sigma_0, i\sigma_1, i\sigma_2, i\sigma_3 \}$ is a basis of the real vector space $u(2)$.

$$\begin{pmatrix} ix & a+ib \\ -a+ib & iy \end{pmatrix} = \frac{x+y}{2} i\sigma_0 + b\, i\sigma_1 + a\, i\sigma_2 + \frac{x-y}{2} i\sigma_3 .$$

Remark: $\sigma_0, \ldots, \sigma_3$ form complex basis of $gl(2,\mathbb{C})$.

For $A \in U(p,q)$ one has $\underbrace{\det_{\mathbb{C}}(A) \underbrace{\det_{\mathbb{C}}(\bar{A}^t)}_{\det_{\mathbb{C}}(A)}}$ $= 1$

$\Longrightarrow |\det_{\mathbb{C}}(A)|^2 = 1$

$\underline{\det_{\mathbb{C}} : U(p,q) \longrightarrow U(1)} \longleftarrow$

$\Longrightarrow SU(p,q) = \{ A \in U(p,q) : \det_{\mathbb{C}}(A) = 1 \}$

is a real Lie ~~sub~~group of dim $n^2 - 1$.

$su(p,q) = \{ X \in gl(n,\mathbb{C}) : I_{p,q} X = -\bar{X}^t I_{p,q} , tr(X) = 0 \}$

- V vector space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

$\exists$ a unique upto isomorphism symplectic structure on V (i.e. $\omega_V : V \times V \longrightarrow \mathbb{K}$ non-deg., skew-symm. bilinear form).

We can identify, $(V, \omega_V) \simeq (\mathbb{K}^{2n}, \omega)$

$\uparrow$ standard symplectic form.

$$\omega(x,y) = x^t J_n y.$$

$$Sp(2n, \mathbb{K}) = \{ A \in GL(2n, \mathbb{K}) : \omega(Ax, Ay) = \omega(x,y) \ \forall x, y \in \mathbb{K}^{2n} \}$$
$$= \{ A \in GL(2n, \mathbb{K}), \underline{A^t J_n A = J_n} \}.$$

$$f(A) = A^t J_n A - J_n \qquad f(A)^t = -f(A)$$

$\Longrightarrow$ $Sp(2n, \mathbb{K})$ is a real (complex) Lie group

and $sp(2n, \mathbb{K}) = \ker(T_{Id} f) = \{ X \in gl(2n, \mathbb{K})$
$: J_n X + X^t J_n = 0 \}$

$\dim_{\mathbb{K}} (sp(2n, \mathbb{K})) = 2n^2 + n$ $\qquad \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$ $B = B^t$
$C = C^t$.

$$U(n) = O(2n) \cap GL(n, \mathbb{C}) \cap Sp(2n, \mathbb{R})$$
$$= O(2n) \cap GL(n, \mathbb{C}) = O(2n) \cap Sp(2n, \mathbb{R})$$
$$= GL(n, \mathbb{C}) \cap Sp(2n, \mathbb{R}).$$

$\longrightarrow$ Kähler manifold.

2. G Lie group

1. $R_x = \nu^* L_{-x} \quad \forall x \in \mathfrak{g}$.

$(gh)^{-1} = g^{-1}h^{-1} \iff \nu \circ \rho^h = \underline{\lambda_{h^{-1}} \circ \nu}$

If $\xi \in \mathfrak{X}_L(G)$, then $\underline{\nu^*\xi} \in \mathfrak{X}_R(G)$.

$(\rho^h)^* \underline{\nu^*\xi} = (\underline{\nu \circ \rho^h})^* \xi = (\lambda_{h^{-1}} \circ \nu)^* = \nu^* \underline{\lambda_{h^{-1}}^*} \xi = \underline{\nu^*\xi}$

$\forall h \in G$.

$\nu^*\xi(e) = (T_e\nu)^{-1}\xi(e) = -\xi(e) \qquad \Leftarrow \xi = L_{\xi(e)}$

(2) follows from (1) $\qquad [L_X, L_Y] = L_{[X,Y]}$

$$[R_X, R_Y] = [\nu^* L_{-X}, \nu^* L_{-Y}] = \nu^* [L_{-X}, L_{-Y}]$$

$$= \nu^* (L_{[X,Y]}) = -R_{[X,Y]}.$$

(3) $(0, L_X)$ is a vector field on $G \times G$

which is μ-related to L_X. $\qquad T_e l_h X$

$$\& \quad T_{(g,h)}\mu\,(0_g, L_X(h)) = T_h l_g L_X(h) = T_e(l_{gh})X$$

$$= L_X(gh).$$

Similarly, for $(R_Y, 0)$ and R_Y.

$\Longrightarrow 0 = \underline{[(0, L_x), (R_y, 0)]}$ is μ-related to $[L_x, R_y]$

Since μ is surjective, this shows $[L_x, R_y](g) = 0$ $\forall g \in G$.