DIFFERENTIAL GEOMETRY

STRUCTURE OF THE COURSE

I. LIE GROUPS

- Basic Theory
- Representations of Lie groups
- Classification of Lie groups
- Homogeneous spaces, Klein geometries

II. BUNDLES

- Fiber bundles, Vector bundles and Principal bundles
- Associated vector bundles
- Homogeneous vector bundles

III. CONNECTIONS

- Linear connections on vector bundles (e.g. affine connections),
- Principal connections on principal connections
- Geometric structures determining (classes) of distinguished affine connections (e.g. Riemannian mfds, conformal mfds.,

projective structures ...)

(• Gauss – Bonnet Thm.)

- Holonomy groups
- Cartan connections, Cartan geometries.

For references / literature: see IS → study material.

<u>Evaluation</u>:

- Homeworks (every other week)
- Exam. (oral / written or both ?).

I. LIE GROUPS

(Čap, Lie groups (lecture notes)).

1.1. Basic Theory

For a group G we write:

- $\mu : G \times G \to G$ for the multiplication map ($\mu(g,h) =: gh \quad \forall g,h \in G$).
- $\nu : G \to G$ for the inversion $\nu(g) = g^{-1}$
- $e \in G$ for the identity / neutral element.

$g \cdot g^{-1} = g^{-1} \cdot g = e$

$e \cdot g = g \cdot e = e \quad \forall g \in G$

Def 1.1 A topological group is a topological space G equipped with a group structure (μ, ν, e) s.t. μ and ν are continuous.

Remark Any abstract group can be made into a topolog. group by equipping it with the discrete topology.

Def. 1.2 A Lie group is a smooth manifold G equipped with a group structure (μ, ν, e) s.t. μ and ν are smooth.

Remark In Def. 1.2 it is enough to require that μ is smooth, since ν is then automatically smooth by applying the Implicit Fct. Thm. to the equation $\mu(g, \nu(g)) = e$.

Def. 1.3

(1) A homomorphism between topolog. groups (resp. Lie groups) G and H is a continuous (resp. smooth) map $\varphi : G \to H$ that is also group homomorphism ($\varphi(gh) = \varphi(g)\varphi(h) \quad \forall g, h \in G$).

(2) A homomorphism $\varphi : G \to H$ as in (1) is called an isomorphism of topolog. groups (resp. Lie groups), if φ is a homeomorphism (resp. diffeomorphism).

Note that in this case φ^{-1} is also a group homomorphism.

<u>Notation</u> Two Lie groups G and H are called isomorphic if $\exists$ a Lie group isomorphism between them. We write $G \simeq H$ in this case.

Groups of greatest interest in mathem. and physics consist of bijections $f: M \longrightarrow M$ of a set M to itself with group multiplication given by composition

$$\mu(f, \tilde{f}) := f \circ \tilde{f} \qquad f, \tilde{f} \in \mathrm{Bij}(M) := \text{set of bijections of } M$$

Inverse $\nu(f) = f^{-1}$ and $e = \mathrm{Id}_M$.

<u>Examples</u> (topolog. groups).

If M has some extra structure, one can consider subgroups of $(\mathrm{Bij}(M), \circ)$ consisting of bijections preserving the extra structure.

- Suppose M is a topolog. space

$$\mathrm{Homeo}(M) := \{ f : M \to M \ : \ f \text{ is a homeomorphism} \}$$

- Suppose M is a smooth mfd. (resp. a smooth oriented mfd.)

$$\mathrm{Diff}(M) := \{ f : M \to M \ : \ f \text{ is a diffeomorphism} \}.$$

$\mathrm{Diff}_+(M) = \{ f : M \to M : f$ is a orientation preserving diffeom.$\}$

- Suppose M is a smooth mfld. equipped with a geometric structure like a Riemannian metric g or a symplectic form ω.

$$\mathrm{Isom}(M,g) := \{ f : M \to M : f \text{ diffeom. s.t. } f^* g = g \}$$
$$\mathrm{Symp}(M,\omega) := \{ f : M \to M : f \text{ diffeom. s.t. } f^* \omega = \omega \}.$$

With the exception of $\mathrm{Isom}(M,g)$, these groups are all infinite-dimensional and cannot be seen as Lie groups

(at least not finite-dimensional ones as we consider).
They are however all naturally topological groups.

$\mathrm{Isom}(M,g)$ is a Lie group of dim. $\leq \frac{\dim(M)(\dim M+1)}{2}$
(see Global Analysis).

Now some examples of actual Lie groups.

<u>Examples</u> (Lie groups)

1. $\mathbb{R}$, $\mathbb{C}$ with respect to $+$ are Lie groups and so is any finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ w.r.t. to $+$

$\mathbb{C}$ and any complex vector space are even complex Lie groups (i.e. holomorphic mfds. with holomorphic group structure).

(2) $\mathbb{R}\setminus\{0\}$, $\mathbb{C}\setminus\{0\}$ are Lie groups w.r.t. to multiplication. (the latter is again a complex Lie group).

Also, $U(1) := S^1 = \{z \in \mathbb{C} : |z| = 1\}$ w.r.t. to multiplication is a Lie group.

(3) The product $G \times H$ of two Lie groups G and H is again a Lie group. In particular, the n-dim. torus $T^n := \underbrace{U(1) \times \dots \times U(1)}_{n}$ is a Lie group.

For $m, n \in \mathbb{Z}_{\geq 0}$ also $\mathbb{R}^m \times T^n$ is a Lie group.

The latter exhaust all connected commutative Lie groups.

(4) If G is a Lie group, then a Lie subgroup H of G is a subgroup of G that is also a submfd.

Since the multipl. on H is just the restriction of the one on G, it is smooth and so H is a Lie group.

(5) Suppose V is a real or complex vector space ($\dim V < \infty$).

$$GL(V) := \{\text{linear isomorphisms of } V\} \subset \operatorname{End}(V) = \{\text{linear maps } V \to V\}.$$

open subset of vector space $\operatorname{End}(V)$.

is a Lie group w.r.t. to composition of linear maps.

(If V is a complex vector space, then $GL(V)$ is a complex Lie group).

Via a choice of basis of V, we can identify $V \cong \mathbb{K}^n$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $GL(V)$ with.

$$GL(n,\mathbb{K}) := \{A \in M_n(\mathbb{K}) : A \text{ invertible}\}$$

$n \times n$ matrices over $\mathbb{K}$

and composition of linear maps becomes matrix multiplication.

It is called the general linear group.

⑥ Matrix groups (also called linear Lie groups) are Lie subgroups of $GL(V)$ resp. $GL(n, \mathbb{K})$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$).

Recall from Global Analysis:

- Special linear group

$$SL(n, \mathbb{K}) := \{ A \in GL(n, \mathbb{K}) : \det_{\mathbb{K}}(A) = 1 \}$$

($SL(n, \mathbb{C})$ is even complex Lie group).

• Orthogonal groups : $I_{p,q} := \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix} \in M_n(\mathbb{R})$

$n = p+q$.

defines standard inner product on $\mathbb{R}^n$ of signature (p,q) :

$$\langle x, y \rangle := x^t I_{p,q}\, y \qquad \forall x, y \in \mathbb{R}^n .$$

$$\begin{aligned} O(p,q) &= \{ A \in GL(n,\mathbb{R}) : A^t I_{p,q} A = I_{p,q} \} \\ &= \{ A \in GL(n,\mathbb{R}) : \langle Ax, Ay \rangle = \langle x, y \rangle \\ &\qquad\qquad \forall x, y \in \mathbb{R}^n \} . \end{aligned}$$

linear orthog. group of sign. (p,q).

$q = 0$: $O(p, 0) = O(n) = \{ A \in GL(n, \mathbb{R}) : A^t = A^{-1} \}$.

$q = 1$: $O(n, 1)$ Lorentzian group,

(= linear isometries of Minkowski space).

- linear symplectic group : $J_n = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} \in M_{2n}(\mathbb{R})$

defines a symplectic structure on $\mathbb{R}^{2n}$:

$$\omega(x, y) = x^t J_n y \qquad \forall x, y \in \mathbb{R}^{2n}.$$

$$Sp(2n, \mathbb{R}) = \{ A \in GL(2n, \mathbb{R}) : A^t J_n A = J_n \}$$
$$= \{ A \in \text{---}''\text{---} : \omega(x, y) = \omega(Ax, Ay) \, \forall x, y \in \mathbb{R}^{2n} \}$$

Suppose (G, μ, ν, e) is a Lie group.

Then we denote by:

- $\lambda_g : G \to G$ left multiplication by $g \in G$,

  $\lambda_g(h) = \mu(g, h) = gh \quad \forall h \in G$.

- $\rho^g : G \to G$ right multiplication by $g \in G$,

  $\rho^g(h) = \mu(h, g) = hg \quad \forall h \in G$.

Lemma 1.4.

For any $g \in G$, λ_g (resp. ρ^g) is a diffeomorphism with inverse $\lambda_{g^{-1}}$ (resp. $\rho^{g^{-1}}$). In particular, if $U \subseteq G$ is an open neighbourhood of $g \in G$, then $\lambda_h(U)$ (resp. $\rho^h(U)$) is an open neighbhd. of hg (resp. gh).

Moreover, for $g, h \in G$ we have $\lambda_g \circ \lambda_h = \lambda_{gh}$ and $\rho^g \circ \rho^h = \rho^{hg}$.

<u>Proof:</u> λ_g resp. ρ^g can be written as compositions of smooth maps:

$$\begin{tikzcd}
G \arrow[r] & G \times G \arrow[r] & G \\
h \arrow[r, mapsto] \arrow[rr, bend right, "\lambda_g"'] & (g, h) \arrow[r, "\mu"] & G
\end{tikzcd}$$

and similarly for ρ^g: $h \mapsto (h, g) \xrightarrow{\mu} hg$.

Clearly, they are diffeom. with inverses $\lambda_{g^{-1}}$ and $\rho^{g^{-1}}$.

Rest is also clear from definitions.

Lemma 1.5 Suppose G is a Lie group.

① For $g, h \in G$, $\xi \in T_g G$, $\eta \in T_h G$ one has

$$\underline{T_{(g,h)}\mu}(\xi, \eta) = \underline{T_h \lambda_g \eta} + \underline{T_g \rho^h \xi}.$$

$$\Big(\mu : G \times G \to G \qquad T\mu : \underbrace{T(G \times G)}_{\simeq TG \times TG} \to TG \Big)$$

② For any $g \in G$,

$$T_g \nu = -T_e \rho^{g^{-1}} \circ T_g \lambda_{g^{-1}} = -T_e \lambda_{g^{-1}} \circ T_g \rho^{g^{-1}}.$$

In particular, $T_e \nu : T_e G \to T_e G$ equals $-\mathrm{Id}_{T_eG}$.

Proof.

① $T_{(g,h)}\mu : \underbrace{T_g G \times T_h G}_{= T_{(g,h)}(G\times G)} \longrightarrow T_{gh} G$ is linear map.

$$\Longrightarrow T_{(g,h)}\mu(\xi,\eta) = T_{(g,h)}\mu(\xi, 0) + T_{(g,h)}\mu(0,\eta).$$

Let $c:(-\varepsilon,\varepsilon)\longrightarrow G$ be a curve repres. ξ

(i.e. $c(0) = g$, $c'(0) = \xi$).

Then $t \longrightarrow (c(t), h)$ represent the tangent vector $(\xi, 0) \in T_{(g,h)}(G\times G)$

$$T_{(g,h)}\mu(\xi,0) = \frac{d}{dt}\Big|_{t=0} \mu(c(t), h) = \frac{d}{dt}\Big|_{t=0} \rho^h(c(t)) =$$
$$= T_g \rho^h \xi$$

Similarly, $T_{(g,h)}\mu(0,\eta) = \frac{d}{dt}\Big|_{t=0} \mu(g, c(t)) = \frac{d}{dt}\Big|_{t=0} \lambda_g(c(t)) =$

$= T_h \lambda_g \eta$

where $c : (-\varepsilon, \varepsilon) \to G$ is a curve repres. η

$(c(0) = h, c'(0) = \eta)$.

② $e = \mu(g, \nu(g))$ $\qquad$ $G \longrightarrow G$, $\quad g \mapsto (g, \nu(g)) \xrightarrow{\mu} G$

Differentiating $\Longrightarrow$ $0 = T_{(g,g^{-1})}\mu\,(T_g \mathrm{id}\,\xi, T_g \nu\,\xi)$

$\overset{①}{=} \underline{T_g \rho^{g^{-1}}}\,\xi + \underline{T_{g^{-1}}\lambda_g T_g \nu}\,\xi \qquad \forall \xi \in T_g G.$

$$\Longrightarrow \quad T_{g^{-1}}\lambda_g T_g \nu\, \xi = -T_g \rho^{g^{-1}} \xi \iff T_g \nu\, \xi = -T_e \lambda_{g^{-1}} \circ T_g \rho^{g^{-1}} \xi$$

Second formula follows similarly from diff. $e = \mu(\nu(g), g)$.

$\square$.