Last week :

G Lie group

A principal G-bundle $\rightleftharpoons$ fiber bundle $p : P \longrightarrow M$ with stand. fiber G and structure gr. G (acting on itself by left multpl.).

There is a right-action of G on P :

$$r : P \times G \longrightarrow P$$

- it restricts to an action on the fibers : $r : \underline{P_x} \times G \longrightarrow P_x$ which is free and transitive

$\Longrightarrow$ $u \in P_x$ induces a diffeom. $G \xrightarrow{\simeq} P_x$ $\quad g \longmapsto u \cdot G = P_x$

Prop. 2.11 Suppose $p: P \to M$ is a smooth surj. map between mflds. and $r: P \times G \to P$ a smooth right action of G that preserves the fibers of p and acts transitively and freely on each fiber.

Then $p: P \to M$ is a principal G-bundle $\Longleftrightarrow$ p admits local smooth sections. In particular, if $p: P \to M$ is a fiber bundle with a smooth fiber-preserving right-action of G on P that is transitive and free on each fiber, then $p: P \to M$ is a principal G-bundle.

Proof:

"$\Rightarrow$" ✓ Choose an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of M so that there is a local section $s_\alpha : U_\alpha \to P$ of p.

Then $\phi_\alpha^{-1} : U_\alpha \times G \longrightarrow p^{-1}(U_\alpha)$ $(*)$

$$(x, g) \longmapsto \underline{s_\alpha(x) \cdot g}$$

is the inverse a a smooth local trivialization $\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G$.

Note that $(*)$ is a bijection, since $P_x \times G \longrightarrow P_x$ is free and transitive $\forall x \in M$.

For any $x \in U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$ (for $\alpha, \beta \in I$), $\exists!\ \psi_{\beta\alpha}(x) \in G$ s.t. $s_\alpha(x) = s_\beta(x) \cdot \psi_{\beta\alpha}(x)$. Implicit fct. Thm. implies that $\psi_{\beta\alpha} : U_{\alpha\beta} \to G$ is smooth (see also Section 2.5).

$$\underline{\phi_\beta \circ \phi_\alpha^{-1}}(x, g) = \phi_\beta(s_\alpha(x)\cdot g) = \phi_\beta(\underline{s_\beta(x)\,\psi_{\beta\alpha}(x)\cdot g})$$

$$= \underline{(x, \psi_{\beta\alpha}(x)\cdot g)}.$$

$\Longrightarrow \{(U_\alpha, \phi_\alpha)\}_{\alpha\in I}$ is a principal G-bundle atlas for $p: P \to M$. $\square$.

Def. 2.12 Suppose $\psi: H \to G$ is a Lie group homomorphism between Lie groups H and G and $q: Q \to N$ and $p: P \to M$ a principal H- (resp. G-) bundle.

① Then a morphism of principal bundles over ψ between q and p is a fiber bundle morphism $f: Q \to P$ s.t.

$$f(u\cdot h) = f(u)\cdot\psi(h) \quad \forall u\in Q, \ \forall h\in H.$$

② Special case of ①: H is a closed subgroup of G and $\varphi = i : H \hookrightarrow G$ is the inclusion.

Then a reduction of structure group to H of a principal G-bundle $p : P \to M$ is a principal H-bundle $q : Q \to M$ together with a morphism of principal bundles $f : Q \to P$ over $i : H \hookrightarrow G$ which covers the identity on M (i.e. $\underline{f} = id_M$).

$$\begin{tikzcd} Q \arrow[rr, "f"] \arrow[dr, "q"'] & & P \arrow[dl, "p"] \\ & M & \end{tikzcd}$$

$$f(u.h) = f(u) \cdot i(h)$$
$$= f(u) \cdot h$$

Note that f is injective $Q \hookrightarrow P$.

③ A morphism of a principal G-bundle $p : P \to M$ to itself covering the identity :

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "p"'] & & P \arrow[dl, "p"] \\ & M & \end{tikzcd}$$

is called a gauge transformation.

(gauge theories in physics are formulated in terms of principal bundles and principal connections).

Lemma 2.13 $p: P \to M$, $q: Q \to N$ principal bundles.

① If $f: P \to Q$ is a morphism of principal bundles and $u \in P$, then $f(u)$ determines the value of f on $P_{p(u)}$.

② Any morphism $f: P \to Q$ of principal G-bundles s.t. $\underline{f}: M \to N$ is a diffeom. is an isomorphism of principal bundles.

In particular, any gauge transformation is an isomorphism of principal G-bundles.

Proof.

① $u' \in P_{p(u)} \implies u' = u \cdot g$ for a unique $g \in G$.

str. gr. of P str. gr of P

$$f(u') = f(u \cdot g) = \underline{\underline{f(u) \cdot \psi(g)}}$$

$\psi : G \to H$ Lie gr. homom.

②
$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\varphi"'] & Q \arrow[d, "q"] \\ M \arrow[r, "\underline{f}"] & N \end{tikzcd}$$

$f|_{P_x} : P_x \to Q_{\underline{f}(x)}$ is bijective, since $f(ug) = f(u) \cdot g)$.

Since $\underline{f}$ is bijective, this implies f is bijection.

Remains to show that f^{-1} is smooth, since then it is automatically a morphism of fiber bundles and G-equivariancy of f implies G-equivariancy of f^{-1}.

Let $\phi : p^{-1}(U) \to U \times G$ and $\psi : q^{-1}(V) \to V \times G$ be principal fiber bundle charts for p and q with $\underline{f}(U) = V$.

Then,
$$\underline{(\psi \circ f \circ \phi^{-1})}(x, g) = \underline{(\underline{f}(x), \mu(x) \cdot g)} \text{ for some}$$
smooth map $\mu : U \to G$.

$\Longrightarrow$ $y \longmapsto \mu(\underline{f}^{-1}(y))^{-1}$ is smooth, since $\underline{f}$ is a diffeo and inversion in G is smooth.

$\Longrightarrow$ $\phi \circ f^{-1} \circ \psi^{-1}$ must be equal to

$$(y, g) \longmapsto \left(\underline{f}^{-1}(y), \mu(\underline{f}^{-1}(y))^{-1} \cdot g \right)$$

which is also smooth.

$\square$.

Examples

① $H \subseteq G$ closed subgr. of a Lie gr. G

Then $p : G \longrightarrow G/H$ is a principal H-bundle.

② $\mathbb{V}$ = real vector space of dim n

$P = \{ \text{linear isomorphisms } u : \mathbb{R}^n \to \mathbb{V} \} \longrightarrow M = \{pt\}$.

principal = $GL(n,\mathbb{R})$ - bundle over a point

principal right-action is given by : $u \in P$, $A \in GL(n,\mathbb{R})$

$$u \cdot A = u \circ A : \mathbb{R}^n \to \mathbb{V} \in P$$

$$P \times GL(n,\mathbb{R}) \longrightarrow P$$
$$(u, A) \longmapsto u \circ A$$

$u, v \in P$, then $A := u^{-1} \circ v : \mathbb{R}^n \to \mathbb{R}^n \in GL(n,\mathbb{R})$.

and $u \cdot A = v$. (i.e, action is transitive)

The action is also free , since $u \circ A = u \longrightarrow A = u^{-1} \circ u = id_{\mathbb{R}^n}$.

Equivalently, elements of P can be identified with the set basis for $\mathbb{V}$.

$$\underbrace{\{u : \mathbb{R}^n \to \mathbb{V}}_{} \longmapsto \underbrace{\{u(e_1), \dots, u(e_n)\}}_{\text{basis of } \mathbb{V}}$$

G

$\{e_1, \dots, e_n\}$ standard basis. $\mathbb{R}^n$

$\{v_1, \dots, v_m\}$

③ M n-dim. mfd., $x \in M$.

$$\begin{tikzcd} TM \arrow[d, "\pi_M"'] \\ M \arrow[u, bend right, "x \mapsto s(x)"'] \end{tikzcd}$$

$$\underline{\tilde{F}_x(M)} := \underline{\{\text{linear isomorphism } \mathbb{R}^n \to T_xM\}} \cong \{\underline{\text{bases of } T_xM}\}$$

$$\tilde{F}(M) := \bigsqcup_{x \in M} \underline{\tilde{F}_x(M)}$$

$$\begin{tikzcd} \tilde{F}(M) \arrow[d, "p"'] \\ M \arrow[u, bend right, "s"'] \end{tikzcd}$$

$M \supset U$

This is a principal $GL(n, \mathbb{R})$-bundle with the principal right action given by $u \in \tilde{F}_x(M)$, $A \in GL(n, \mathbb{R})$

$u \cdot A = u \circ A$.

$GL(n,\mathbb{R})$-atlas : T_xM n

$\mathbb{R}^n \xrightarrow{\sim} T_xM$ n^2

$u_\alpha : U_\alpha \to u_\alpha(U_\alpha) \subseteq \mathbb{R}^n$

(U_α, u_α) chart of M. $\mathbb{R}^{n*} \otimes T_xM$

$T_x u_\alpha : \underset{= T_xM}{T_x U_\alpha} \xrightarrow{\sim} \mathbb{R}^n$

$$\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times GL(n,\mathbb{R})$$

$v : \mathbb{R}^n \to T_{p(v)}M$

$$\phi_\alpha(v) := \left(p(v), T_{p(v)}u_\alpha \circ v\right)$$

$$\phi_\beta \circ \phi_\alpha^{-1}(x, A) = \Big(x, \underbrace{T_{u_\alpha^{-1}(x)} u_\beta \circ T_x u_\alpha^{-1}}_{\overset{!!}{=}\ \psi_{\beta\alpha}(x)} \circ A\Big) = \left(x, \underline{\psi_{\beta\alpha}(x) \circ A}\right)$$

$\psi_{\beta\alpha} : U_{\alpha\beta} \to GL(n,\mathbb{R})$ smooth.

By Prop. 2.4, $\mathcal{F}(M) \to M$ is a principal $GL(n,\mathbb{R})$-bundle, called the (linear) frame bundle of M.

Note that any local section $s: U \to \mathcal{F}(M)$ of ~~the~~ $\mathcal{F}(M)$ gives rise to a local frame of TM defined on U. & and conversely.

(4) Suppose $V \to M$ vector bundle with standard fiber $\mathbb{V}$.

$$\mathcal{F}(V) := \bigsqcup_{x \in M} \mathcal{F}_x(V) \qquad \mathcal{F}_x(V) = \{\text{linear isom. } \mathbb{V} \xrightarrow{\cong} V_x\}$$

$$\downarrow \qquad\qquad\qquad\qquad \cong \{\text{basis of } V_x\}.$$

M

~~Bases~~ called **the frame bundle of V.**

is a principal $GL(\mathbb{V})$-bundle ; $u \in \mathcal{F}_x(V)$, $A \in GL(\mathbb{V})$

$u \cdot A = u \circ A$.

Vector bundle charts for V give rise to principal bundle chart for $\mathcal{F}(V)$

Local (smooth) sections of $\mathcal{F}(V) \rightleftharpoons$ Local frames of V.

$(\mathcal{F}(TM) = \mathcal{F}(M))$.