HOMEWORK 2

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$$

① Cross product on $\mathbb{R}^3$: $\times : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$v \times w = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} \in \mathbb{R}^3 \qquad \forall v, w \in \mathbb{R}^3$$

Evidently, $v \times w = - w \times v \quad \forall v, w \in \mathbb{R}^3$

$\lambda \in \mathbb{R}$, $\lambda v \times w = \lambda (v \times w) = v \times \lambda w$

$$(v + u) \times w = \begin{pmatrix} (v_2 + u_2) w_3 - (v_3 + u_3) w_2 \\ (v_3 + u_3) w_1 - (v_1 + u_1) w_3 \\ (v_1 + u_1) w_2 - (v_2 + u_2) w_1 \end{pmatrix} = \begin{pmatrix} v_2 w_2 - v_3 w_3 \\ \vdots \\ \end{pmatrix} + \begin{pmatrix} u_2 w_3 - u_3 w_2 \\ \vdots \\ \end{pmatrix}$$

$$= v \times w + u \times w$$

Jacobi identity: $u \times (v \times w) + \underline{v \times (w \times u)} + w \times (u \times v) = 0$

$$\begin{pmatrix} u_2(\cancel{v_1 w_2} - \cancel{v_2 w_1}) - u_3(\cancel{v_3 w_1} - \cancel{v_1 w_3}) \\ \vdots \\ \vdots \end{pmatrix} + \begin{pmatrix} v_2(\cancel{w_1 u_2} - \cancel{w_2 u_1}) - v_3(\cancel{w_3 u_1} - \cancel{w_1 u_3}) \\ \vdots \\ \vdots \end{pmatrix}$$

$$+ \begin{pmatrix} w_2(\cancel{u_1 v_2} - \cancel{u_2 v_1}) - w_3(\cancel{u_3 v_1} - \cancel{u_1 v_3}) \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Longrightarrow$ $(\mathbb{R}^3, \times)$ is a 3-dim. Lie algebra.

$$\mathfrak{so}(3,\mathbb{R}) = \{ X \in M_3(\mathbb{R}) : X^t = -X \} =$$

$$= \left\{ \begin{pmatrix} 0 & -x_1 & -x_2 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{pmatrix} : x_i \in \mathbb{R} \right\}$$

$\Phi : \mathfrak{so}(3,\mathbb{R}) \longrightarrow \mathbb{R}^3$ linear isomorphism.

$$\begin{pmatrix} 0 & -x_1 & -x_2 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \phi([X,Y]) = \phi(X) \times \phi(Y)$$

$$\left[\begin{pmatrix} 0 & -x_1 & -x_2 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -y_1 & -y_2 \\ y_1 & 0 & y_3 \\ y_2 & y_3 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & - & \cdot \\ x_2 y_3 - x_3 y_2 & 0 & - \\ x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 & 0 \end{pmatrix}$$

② $SL(2,\mathbb{K})$ $\quad \mathbb{K} = \mathbb{R}, \mathbb{C}$ $\quad , \ \mathfrak{sl}(2,\mathbb{K})$

$$\mathfrak{sl}(2,\mathbb{K}) = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} : x_i \in \mathbb{K} \right\} \qquad \dim_{\mathbb{K}}(\mathfrak{sl}(2,\mathbb{K})) = 3$$

$$\langle \ , \ \rangle : \mathfrak{sl}(2,\mathbb{K}) \times \mathfrak{sl}(2,\mathbb{K}) \longrightarrow \mathbb{K}$$

$$\langle X, Y \rangle := \frac{1}{2} \underline{\mathrm{tr}(XY)}$$

- $\mathrm{tr}(XY) = \sum_{i=1}^{3} x_i y_i = \sum_{i=1}^{2} y_i x_i = \mathrm{tr}(XY)$
- $\mathrm{tr}((X + \lambda Z) Y) = \mathrm{tr}(XY + \lambda ZY) = \mathrm{tr}(XY) + \lambda \, \mathrm{tr}(ZY)$

$X, Y, Z \in \mathfrak{gl}(2,\mathbb{K})$
$\lambda \in \mathbb{K}$

$\Longrightarrow \langle \ , \ \rangle$ is a symmetric bilinear form.

Non-degeneracy: Suppose $X \in \mathfrak{sl}(2,\mathbb{K})$ s.t. $\underline{\operatorname{tr}(XY) = 0}$ $\forall Y \in \mathfrak{sl}(2,\mathbb{K})$

In particular, $\underline{0} = \operatorname{tr}\left(X \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \operatorname{tr}\begin{pmatrix} x_1 & -x_2 \\ x_3 & x_1 \end{pmatrix} = \underline{2x_1}$

$$0 = \operatorname{tr}\left(X \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = x_2$$

$$0 = \operatorname{tr}\left(X \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = x_3 \qquad X = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix}$$

$\Longrightarrow X = 0$

It has signature $(2,1)$ on $\mathfrak{sl}(2,\mathbb{R})$: $\underline{E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \quad E_3 = \begin{pmatrix} 0 & 1 \\ \underline{-1} & 0 \end{pmatrix}$

E_1, E_2, E_3 is an orthog. basis of $(\mathfrak{sl}(2,\mathbb{R}), \langle\, , \rangle)$.

$\langle E_1, E_1 \rangle = \frac{1}{2} \operatorname{tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \qquad \langle E_2, E_2 \rangle = \frac{1}{2} \operatorname{tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

$$\langle E_3, E_3 \rangle = \frac{1}{2} \operatorname{tr}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1 .$$

<u>Remark</u>: No signature over $\mathbb{C}$.

Over $\mathbb{R}$: $S \in M_n^{sym}(\mathbb{R})$ $\exists A \in GL(n, \mathbb{R})$

s.t. $A S A^t = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & -1 & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix}$.

Over $\mathbb{C}$: $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ Lorentzian ~~metric~~ metric on $\mathbb{R}^2$

$$A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} a_{11}^2 & 0 \\ 0 & -a_{22}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\cancel{A}$ e $A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$ $\qquad a_{11} = \pm 1, \; a_{22} = \pm i$

- $\underline{\mathrm{Ad}} : SL(2,\mathbb{K}) \longrightarrow GL(\underline{\mathfrak{sl}(2,\mathbb{K})}) \simeq GL(3,\mathbb{K})$

$$\mathrm{Ad}(A)(X) = AXA^{-1},$$

$$\underline{\langle \mathrm{Ad}(A)X, \mathrm{Ad}(A)Y \rangle} = \langle AXA^{-1}, AYA^{-1} \rangle$$

$$= \frac{1}{2}\operatorname{tr}(AX\underbrace{A^{-1}A}_{\mathrm{Id}}YA^{-1}) = \frac{1}{2}\operatorname{tr}(A\underline{XY}A^{-1})$$

$$= \frac{1}{2}\operatorname{tr}(XY) = \underline{\langle X,Y \rangle}$$

$$\mathrm{Ad}(A) \in O(\mathfrak{sl}(2,\mathbb{K}), \langle\, , \rangle) \quad \forall A \in SL(2,\mathbb{K}).$$

$$O(\mathfrak{sl}(2,\mathbb{C}), \langle\, , \rangle) \simeq O(3,\mathbb{C})$$

$$O(\mathfrak{sl}(2,\mathbb{R}), \langle\, , \rangle) \simeq O(2,1)$$

Since $SL(2,\mathbb{K})$ is connected, $Ad(SL(2,\mathbb{K}))$ is a connected subgroup of $O(3,\mathbb{C})$ resp. $O(2,1)$ containing the identity.

$\Rightarrow$ Ad has values in $SO(3,\mathbb{C})$ resp. $SO_0(2,1)$.

$$T_{id}Ad = ad : \overset{3}{\mathfrak{sl}(2,\mathbb{C})} \longrightarrow \overset{3}{\mathfrak{so}(3,\mathbb{C})}$$
$$\mathfrak{sl}(2,\mathbb{R}) \longrightarrow \mathfrak{so}(2,1)$$

$$\underset{\underset{[X,Y]}{\shortparallel}}{ad(X)(Y)} = 0 \quad \forall\, Y \in \mathfrak{sl}(2,\mathbb{K}). \qquad \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix}$$

$$0 = ad(X)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2\begin{pmatrix} 0 & -x_2 \\ x_3 & 0 \end{pmatrix} \longrightarrow x_2 = 0$$

$$0 = ad(X)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x_2 & 0 \\ -2x_1 & -x_2 \end{pmatrix} \Longrightarrow x_1 = 0 \quad x_3 = 0$$

$\Longrightarrow \ker \mathrm{ad} = 0$

$\Longrightarrow$ $\mathrm{ad} : \mathfrak{sl}(2,\mathbb{C}) \xrightarrow{\sim} \mathfrak{so}(3,\mathbb{C})$ is omomorphism.

$\mathfrak{sl}(2,\mathbb{R}) \xrightarrow{\sim} \mathfrak{so}(2,1)$

$\Longrightarrow$ $\mathrm{Ad} : SL(2,\mathbb{C}) \twoheadrightarrow SO(3,\mathbb{C})$ are covering maps.

$SL(2,\mathbb{R}) \twoheadrightarrow SO_0(2,1)$

with kernel. $\ker(\mathrm{Ad}) = \{\pm \mathrm{Id}\}$

$\mathrm{Ad}(A)(X) = X \quad \forall X \in \mathfrak{sl}(2,\mathbb{K})$

$\quad \|$

AXA^{-1}

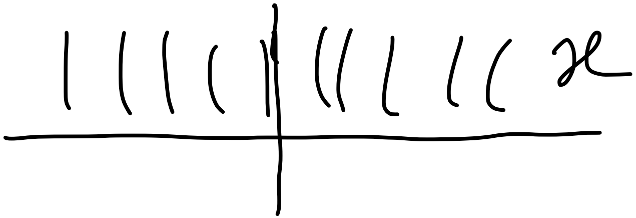
$AX = XA$

$\Longrightarrow A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad a \in \mathbb{K}$

$\det_{\mathbb{K}}(A) = a^2 = 1$

$\Longrightarrow a = \pm 1$

(3) $\mathcal{H} = \{(x,y)\in\mathbb{R}^2 : y>0\} = \{z\in\mathbb{C} : \mathrm{Im}(z)>0\}$



$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R}) : ad-bc=1 \right\} \Rightarrow (c,d)\neq(0,0)$$

$$SL(2,\mathbb{R}) \times \mathcal{H} \to \mathcal{H} \qquad (*)$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \longmapsto \frac{az+b}{cz+d} \qquad \text{(Möbius transformation)}$$

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\mathrm{Im}(z)}{|cz+d|^2} > 0$$

$\Longrightarrow$ $(*)$ is well-def. and smooth.

. It is a group action:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z = \frac{z}{1} = z \quad \forall z \in \mathcal{H}.$$

$$\left(\begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \cdot z = \frac{(ea+fc)z + eb + fd}{(ga+hc)z + gb + hd}$$

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix} \cdot \left(\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z}_{\frac{az+b}{cz+d}} \right) = \frac{e\left(\frac{az+b}{cz+d}\right) + f}{g\left(\frac{az+b}{cz+d}\right) + h} = \frac{\frac{eaz + eb + fcz + fd}{cz+d}}{\frac{gaz + gb + hcz + hd}{cz+d}}$$

$$= \frac{(ea+fc)z + eb + fd}{(ga+hc)z + gb + hd}$$

$\Longrightarrow$ (*) is a smooth group action.

Transitive: i, $z = x + iy$ $\quad y > 0$

$$\begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \cdot i = \frac{\sqrt{y}\, i + \frac{x}{\sqrt{y}}}{\frac{1}{\sqrt{y}}} = yi + x = z$$

$\in SL(2,\mathbb{R})$

$\underline{SL(2,\mathbb{R})i = \mathcal{H}}$, $\quad w \in \mathcal{H} \quad \exists\, A \in SL(2,\mathbb{R})$ s.t.

$A \cdot i = w \quad \Longrightarrow \quad i = A^{-1}w$

$\Longrightarrow \exists\, B \in SL(2,\mathbb{R})$ s.t. $Bi = z$

$\Longrightarrow B \cdot A^{-1} \in SL(2,\mathbb{R}) \quad BA^{-1}w = z$.

Isotropy group of $i \in \mathcal{H}$:

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\cdot i = i$$

$$\frac{ai+b}{ci+d} = i \iff \underline{ai+b} = i(ci+d) = \underline{-c+id}$$

$$\iff b = -c \text{ and } d = a .$$

Isotropy group of $i = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} , a^2+b^2 = 1 \right\}$

$$= \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}$$

$$\simeq SO(2)$$

$$\implies \mathcal{H} \simeq SL(2,\mathbb{R})/SO(2)$$

- $g = \dfrac{dx^2 + dy^2}{y^2} = \dfrac{4\,|dz|^2}{|z - \bar{z}|^2}$

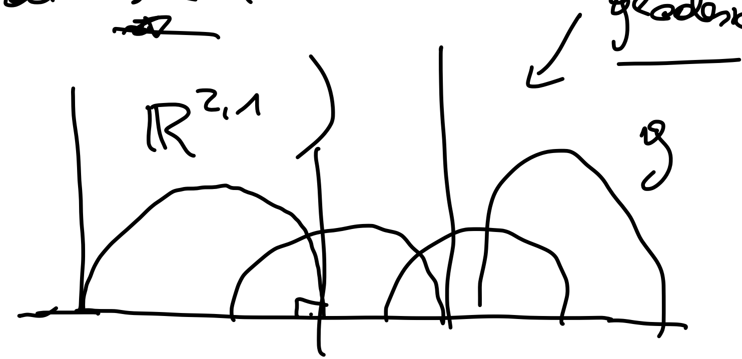
$z = x + iy$

$dz = dx + i\,dy$.

defines a Riemannian metric on $\mathcal{H}$.

(2-dim hyperbolic space. $\mathbb{R}^{2,1}$)

$O_+(2,1)$



For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$

$$z \longmapsto \frac{az+b}{cz+d} = w(z)$$ diffeom. $\mathcal{H} \to \mathcal{H}$

with inverse $w \longmapsto z = \dfrac{dw - b}{-cw + a}$

$$\underline{dw} = w'(z)\, dz = \left(\frac{a}{cz+d} - \frac{c(az+b)}{(cz+d)^2} \right) dz$$

$$= \frac{a(\cancel{cz}+d) - c(\cancel{az}+b)}{(cz+d)^2} dz = \frac{ad - bc}{(cz+d)^2} dz = \underline{\frac{1}{(cz+d)^2} dz}$$

$ad - bc \overset{\shortparallel}{} 1$

$$\operatorname{Im}(w) = \frac{\operatorname{Im}(z)}{|cz+d|^2} \qquad \operatorname{Im}(w)^2 = \frac{\operatorname{Im}(z)^2}{|cz+d|^4}$$

$$\operatorname{Im}(w)^2 \shortparallel |w - \overline{w}|^2$$

$$\frac{4\,|dw|^2}{|w-\overline{w}|^2} = \frac{4\left|\frac{1}{(cz+d)^2} dz\right|^2}{\frac{\operatorname{Im}(z)^2}{|cz+d|^4}} = \frac{4\,\frac{1}{|cz+d|^4}\,|dz|^2}{\frac{\operatorname{Im}(z)^2}{|cz+d|^4}} \Rightarrow$$

$$= \frac{4\,|dz|^2}{|z-\overline{z}|^2}$$

(4) G conn. Lie group. $\phi : G \to GL(V)$
$\phi' : \mathfrak{g} \to \mathfrak{gl}(V)$.

. $W \subset V$ is G-inv. $\Leftrightarrow$ it is $\mathfrak{g}$-inv.

$\phi(\exp(tx))(\omega)$ $\parallel$

"$\Rightarrow$": $x \in \mathfrak{g}$, $\omega \in W$, then $t \mapsto \underbrace{\exp(tx)}_{\in G} \cdot \omega \in W$ $\forall t$.

$$\Rightarrow \phi'(x)(\omega) = \frac{d}{dt}\Big|_{t=0} \phi(\exp(tx))(\omega) \in W$$

$\parallel$

$x \cdot \omega$

"$\Leftarrow$": $\phi(\exp(x)) = e^{\phi'(x)} = \sum_{k=0}^{\infty} \frac{1}{k!} \phi'(x)^k$ $\forall x \in \mathfrak{g}$.

Since $\phi'(x)(W) \subset W$ by assumption, also $\phi'(x)^k(W) \subset W$ $\forall k$.

Since $W \subseteq V$ is closed subset of V, we conclude that

$$\phi(\underline{\exp(x)})(W) \subseteq W \qquad \forall x \in \mathfrak{g}.$$

$\Rightarrow$ W is invariant under the group generated by $\exp(\mathfrak{g}) \subseteq G$, which however equals $\underline{G}$ by connectedness of G.

. V is unitary as G-rep. $\Leftrightarrow$ it is as $\underline{\mathfrak{g}}$-rep resp.

$\Rightarrow$ $\phi : G \to O(V)$ resp. $U(V)$

$\Rightarrow$ $\phi' : \mathfrak{g} \to \mathfrak{o}(V)$ resp. $\mathfrak{u}(V)$

$\Leftarrow$ $\phi' : \mathfrak{g} \to \mathfrak{o}(V), \mathfrak{u}(V)$ $\qquad \phi(\underline{\exp(x)}) = \exp(\phi'(x)) \subseteq O(V)$ or $U(V)$

(5) G compact. $\dim(G) = n$.

$$0 \neq \omega \in \Lambda^n \mathfrak{g}^* \Longrightarrow \mathrm{vol}(g) := \omega\left(T_g \lambda_{g^{-1}} -, \ldots, T_g \lambda_{g^{-1}} -\right)$$

$$\int_G f := \int_G f \, \mathrm{vol}$$

- $\lambda_g^* \mathrm{vol} = \mathrm{vol} \quad \forall g \in G$ (λ_g is an orientation preserving diffeom.)

$$\int_G f := \int_G f \, \mathrm{vol} = \int_G \lambda_g^* (f \, \mathrm{vol}) = \int_G f \circ \lambda_g \, \mathrm{vol}$$

$$= \int_G f \circ \lambda_g .$$

- V complex or real repr. of G.

$b : V \times V \to \mathbb{K}$ pos. def. (Hermitian) inner product.

$$\langle v, w \rangle := \int_G \underbrace{b(g^{-1}v, g^{-1}w)}_{=: f_{v,w}}$$

- $\langle v_1 + t v_2, w \rangle = \langle v_1, w \rangle + t \langle v_2, w \rangle$,

- $f_{w,v} = \overline{f_{v,w}}$

- $f_{v,v}(g) > 0$ $\forall g \in G$, $v \neq 0$

$$\langle g^{-1}v, g^{-1}w \rangle = \langle v, w \rangle$$