<u>Thm. 1.29</u> Suppose H is a closed subgroup of a Lie group G. Then H is a Lie subgroup of G.

<u>Proof</u>

$\mathfrak{h} := \{ c'(0) \; : \; c : \mathbb{R} \longrightarrow H \subseteq G \text{ is smooth}, \; c(0) = e \}$

$\subseteq \mathfrak{g}$.

is a linear subspace of $\mathfrak{g}$.

<u>$\mathfrak{h} = \{ X \in \mathfrak{g} \; : \; \exp(tX) \in H \;\; \forall t \in \mathbb{R} \}$</u>.

<u>Claim 4.</u> Write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ as a vector space ($\mathfrak{k}$ is a linear compliment of $\mathfrak{h}$ in $\mathfrak{g}$).

Then $\exists$ an open neighbhd. W of $0 \in \mathfrak{k}$ in $\mathfrak{k}$ s.t.
$\exp(W) \cap H = \{e\}$.

Conversely, assume that's not the case. Then $\exists$ a sequence of elements $\underset{\neq 0}{Y_n} \in \mathfrak{k}$ s.t. $\lim_{n\to\infty} Y_n = 0$ and $\exp(Y_n) \in H$ $\forall n \in \mathbb{N}$.

For a norm $\|.\|$ on $\mathfrak{k}$, put $X_n := \frac{1}{\|Y_n\|} Y_n$.
By passing to a subsequence if necessary, we can assume that $\lim_{n\to\infty} X_n =: X \in \mathfrak{k}$. Then $\|X\| = 1$ (in particular, $X \neq 0$).
Set $t_n := \|Y_n\|$. Then $\exp(t_n X_n) = \exp(Y_n) \in H$ $\forall n \in \mathbb{N}$.

Claim 2 and 3 show that $X \in \mathfrak{g}$, which is a contradiction
$\left(\Leftrightarrow \exp(tX) \in H \; \forall t \in \mathbb{R} \right) \Downarrow X \in \mathfrak{g}$ to $0 \neq X$ and $X \in \mathfrak{k}$.

Define the following C^∞-map:

$$F : \mathfrak{g} \times \mathfrak{k} \longrightarrow G$$

$$F(X, Y) := \exp(X) \cdot \exp(Y) = \mu(\exp(X), \exp(Y))$$

$T_0 F$ is a linear isomorphism, hence $\exists$ open neighbhds. V and W of $0 \in \mathfrak{g}$ and $0 \in \mathfrak{k}$ respect. s.t.

$$F|_{V \times W} : V \times W \longrightarrow F(V, W) =: U$$

is a diffeom. onto an open neighbhd. U of e in G.

By possibly shrinking W, we may assume $\exp(W) \cap H = \{e\}$ by claim 4.

F restricted to $V \times \{0\}$ is a bijection onto $U \cap H$.

Indeed, $\exp(V) \subseteq U \cap H$, since $V \subseteq \mathfrak{h}$. Moreover, any $x \in U \cap H$ can be uniquely written as $x = \exp(X)\exp(Y)$ for $X \in V$ and $Y \in W$. $\Rightarrow$

$$\exp(Y) = \underbrace{\underbrace{\exp(-X)}_{\in H} \underset{\in H}{x}}_{\in H} \in H$$

$\Rightarrow \exp(Y) = e \Rightarrow Y = 0$

$\Longrightarrow$ $\left(U, u := F\Big|_{V\times W}^{-1}\right)$ is a submfd. chart for H defined around $e \in G$ and $\left(\lambda_h(U), u \circ \lambda_{h^{-1}}\right)$ is one for any $h \in H$.

$\Longrightarrow$ $H \subseteq G$ is a smooth submfd.

$\square$.

Examples

① $\varphi : G \to H$ Lie group homomorphism

Then $\ker(\varphi) = \varphi^{-1}(e)$ is a normal subgroup of G, which is closed. Hence, $\ker(\varphi)$ is a Lie subgroup of G.

(2) Center of a group G:

$$Z(G) := \{ g \in G : gh = hg \quad \forall h \in G \}.$$

This subgroup of G.

Note that for any $h \in G$: $f_h : G \longrightarrow G$ is smooth.

$$g \mapsto g^{-1} h^{-1} g h$$

(in particular continuous).

$\Longrightarrow$ $f_h^{-1}(e) = \{ g \in G : gh = hg \} \subseteq G$ is closed.

$\Longrightarrow$ $Z(G) = \bigcap_{h \in G} f_h^{-1}(e) \subseteq G$ is closed,

$\Longrightarrow$ $Z(G)$ is a Lie subgroup of G.

③ Any closed subgroup of $GL(n,\mathbb{R})$ is a Lie subgroup of $GL(n,\mathbb{R})$.

For some purposes the notion of a Lie subgroup of a Lie group is too restrictive:

<u>Def. 1.30</u> Suppose G is a Lie group. Then a virtual Lie subgroup of G is the image of an injective Lie group homomorphism $i: H \to G$.

<u>Prop. 1.31</u> Let G be a Lie group and $i : H \to G$ a virtual Lie subgroup.

① i is an immersion (i.e. $T_h i : T_h H \to T_{i(h)} G$ is injective $\forall h \in H$)

In particular, $i' = T_e i : \mathfrak{h} \to \mathfrak{g}$ is an injective Lie algebra homomorphism.

(Hence, $\mathfrak{h}$ can be identified with ~~as~~ the subalgebra $i'(\mathfrak{h}) \subseteq \mathfrak{g}$).

② Assume G and H are connected. Then

$i(H)$ is normal subgroup $\Leftrightarrow$ $\mathfrak{h}\ (= i'(\mathfrak{h})) \subseteq \mathfrak{g}$ is an ideal (i.e. $[X,Y] \in \mathfrak{h}$ $\forall X \in \mathfrak{h}, Y \in \mathfrak{g}$).

Proof: 1

$i = \lambda_{i(h)} \circ i \circ \lambda_{h^{-1}}$ implies

$i' = T_e i : \mathfrak{h} \to \mathfrak{g}$ is injective $\Longleftrightarrow$ $T_h i : T_h H \to T_{i(h)} G$ is inj. $\forall h \in H$.

By Thm. 1.23 (1), $i(\exp_p^H(tx)) = \exp_p^G(t\, i'(x))$ $\forall x \in \mathfrak{g}$ $\forall t \in \mathbb{R}$.

So, $i'(x) = 0$ implies $\exp_p^H(tx) = e$ $\forall t \in \mathbb{R}$

by injectivity of i. $\Longrightarrow$ $X = 0$.

2 By definition, $i(H) \subseteq G$ is a normal subgroup

$\Longleftrightarrow$ $\mathrm{conj}_g(i(h)) \in i(H)$ $\forall g \in G, \forall h \in H$.

,$\Longrightarrow$' For any $X \in \mathfrak{h}$ and $t \in \mathbb{R}$, $\exp(tX) \in H$

and
$$\mathrm{conj}_g\big(\underbrace{i(\exp(tX))}\big) \underset{\text{Thm. 1.23}}{=} \mathrm{conj}_g\big(\exp(t\, i'(X))\big) =$$
$$\underline{\in i(H)}$$
$$\underset{\text{Prop. 1.26 } \cancel{\text{Thm. 1.23}}}{=} \underline{\exp\big(t \operatorname{Ad}(g)(i'(X))\big)}$$

Differentiating at $t=0$ yields, $\underline{\operatorname{Ad}(g)(i'(X)) \subseteq i'(\mathfrak{h})}$

$\forall X \in \mathfrak{h}, \forall g \in G$.

For $g = \exp(tY)$ for $Y \in \mathfrak{g}$, $t \in \mathbb{R}$,

this gives
$$\operatorname{Ad}(\exp(tY))(i'(\mathfrak{h})) \subseteq i'(\mathfrak{h}) \quad \forall t \in \mathbb{R}, \forall Y \in \mathfrak{g}.$$

Different. at $t=0$ yields:

$$(*) \quad \mathrm{ad}(Y)\big(i'(\mathfrak{h})\big) \subseteq i'(\mathfrak{h}) \qquad \forall Y \in \mathfrak{g}$$

$$\|$$

$$[Y, i'(\mathfrak{h})] \subseteq i'(\mathfrak{h}) \qquad \forall Y \in \mathfrak{g}.$$

i.e. $i'(\mathfrak{h}) \subseteq \mathfrak{g}$ is an ideal.

$$\Leftarrow \text{"} \quad \underline{\mathrm{ad}(Y)(i'(\mathfrak{h})) \subseteq i'(\mathfrak{h})} \qquad \forall Y \in \mathfrak{g}.$$

$$\Longrightarrow \quad \underbrace{\mathrm{Ad}(\underline{\exp}(Y))(i'(\mathfrak{h}))}_{\text{Prop. 1.26}} = \underbrace{e^{\mathrm{ad}(Y)}(i'(\mathfrak{h}))}_{\subseteq i'(\mathfrak{h})}$$

Since G is connected and Ad a group homomorph.,
(2) of Thm. 1.23 shows

that $\underline{\mathrm{Ad}(g)(i'(\mathfrak{h}))} \subseteq i'(\mathfrak{h}) \quad \forall g \in G.$

$\subseteq \mathfrak{g}$

$$\Longrightarrow \exp(\underbrace{\mathrm{Ad}(g)i'(X)}_{\in i'(\mathfrak{h})}) \underset{\text{Prop. 1.26}}{=} \mathrm{conj}_g(i(\exp(X))) \in i(H) \quad \forall X \in \mathfrak{h}.$$

Since H is connected and conj_g a group homomorphism,
this shows that $\mathrm{conj}_g(i(H)) \subseteq i(H) \quad \forall g \in G.$

$\square$.

<u>Def. 1.32</u> Suppose $\mathfrak{g}$ is a Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ an ideal ($[X,Y] \in \mathfrak{h}$ $\forall x \in \mathfrak{g}, Y \in \mathfrak{h}$).

Then the quotient vector space $\mathfrak{g}/\mathfrak{h} = \{X + \mathfrak{h} : X \in \mathfrak{g}\}$ has a natural Lie algebra structure

$$[X+\mathfrak{h}, Y+\mathfrak{h}] := [X,Y] + \mathfrak{h}$$

Check this is well-defined if $\mathfrak{h}$ is an ideal.

Then $(\mathfrak{g}/\mathfrak{h}, [\,,\,])$ is called the quotient of $\mathfrak{g}$ by the ideal $\mathfrak{h}$.

QUESTION: For a Lie group G, is any subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ a Lie algebra of a Lie subgroup of G?

Thm. 1.33 Suppose G is a Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{h} \leq \mathfrak{g}$ be a subalgebra. Then there exists a unique connected virtual Lie subgroup $i: H \to G$ s.t. $i'(T_eH) = \mathfrak{h}$ ($i': T_eH \xrightarrow{\sim} \mathfrak{h}$ isomorphism.).

Moreover, $i(H) \subseteq G$ is an initial submfd.

Proof Left trivialization of TG :

$$\begin{tikzcd} TG \arrow[rr, "\sim"] \arrow[dr, "p"'] & & G \times \mathfrak{g} \arrow[ll, bend right] \arrow[dl, "pr_1"] \\ & G & \end{tikzcd}$$

$\xi_g \in T_g G$

$\xi_g \longmapsto (g, T_g \lambda_{g^{-1}} \xi_g)$

$L_X(g) \longleftarrow (g, X)$

$= T_e \lambda_g X$.

Let $E \subset TG$ be the smooth distribution corresp. to $G \times \mathfrak{h}$ under the left trivialization.

$$E_g := \{ \xi_g \in T_g G \ . \ T_g \lambda_{g^{-1}} \xi_g \in \mathfrak{h} \}$$

$\subseteq T_g G$

Choose a basis $\{X_1, \dots, X_k\}$ of $\mathfrak{h} \subseteq \mathfrak{g}$, then

$\underline{L_{X_1}(g), \dots, L_{X_k}(g)}$ form a basis of E_g.

<u>Claim 1</u> $E \subset TG$ is integrable.

By the Frobenius Thm., it is sufficient to show that E is involutive.

$\xi, \eta \in \Gamma(E)$ (i.e. $\xi_g, \eta_g \in E_g \ \forall g \in G$).

$$\xi(g) = \sum_{i=1}^{k} \underline{\xi_i(g)}\, L_{X_i}(g) \quad \forall g \in G. \qquad \xi_i, \eta_i : G \to \mathbb{R} \text{ are smooth } \forall i.$$

$$\eta(g) = \sum_{i=1}^{k} \underline{\eta_i(g)}\, L_{X_i}(g)$$

$$[\xi,\eta] = \left[\sum_i \xi_i L_{x_i}, \sum_i \eta_i L_{x_i}\right] =$$

$$= \sum_{i,j} \underbrace{[\xi_i L_{x_i}, \eta_j L_{x_j}]}_{\xi_i [L_{x_i}, \eta_j L_{x_j}] - ((\eta_j L_{x_j})\cdot \xi_i) L_{x_i}}$$

$$= \xi_i \eta_j \underbrace{[L_{x_i}, L_{x_j}]}_{= L_{\underbrace{[x_i, x_j]}_{\in \mathfrak{g}}}} + \xi_i (L_{x_i}\cdot \eta_j) L_{x_j} - ((\eta_j L_{x_j})\cdot \xi_i) L_{x_i}$$

$\in \mathfrak{g}$, since $\mathfrak{g}$ is a subalgebra.

$$\Longrightarrow [\xi,\eta](g) \in E_g \quad \forall g \in G.$$

<u>Claim 2.</u> $H := \mathcal{F}_e^E$ leaf of the foliation $\mathcal{F}^E$ corresp. to E through $e \in G$.
$i : H \hookrightarrow G$

is a connected virtual Lie subgroup of G s.t. $i(H)$ is an initial submfd and $i'(T_e H) = \mathfrak{h}$.

From GA-class, we know that $i : \mathcal{F}_e \hookrightarrow G$ is connected initial submfd. of G and $T_e i\,(T_e \mathcal{F}_e^E) = \mathfrak{h}$.

It remains to show that H is a smooth Lie group:

Note that, $E_{gh} = T_h \lambda_g \, E_h \quad \forall g, h \in G$.

$\Longrightarrow$ For $g \in G$, $\lambda_g(\mathcal{F}_e^E) = \mathcal{F}_g^E$.

If $g \in \tilde{\mathcal{F}}_e^E$, then $\tilde{\mathcal{F}}_g^E = \tilde{\mathcal{F}}_e^E$ and hence

$g, h \in \tilde{\mathcal{F}}_e^E$, $gh = \lambda_g(\tilde{\mathcal{F}}_e^E) = \tilde{\mathcal{F}}_g^E = \tilde{\mathcal{F}}_e^E$

$g \in \tilde{\mathcal{F}}_e^E$, $g^{-1} \in \tilde{\mathcal{F}}_e^E$ since $\lambda_g(\tilde{\mathcal{F}}_e^E) = \tilde{\mathcal{F}}_e^E$.

$\Longrightarrow$ $H := \tilde{\mathcal{F}}_e^E \hookrightarrow G$ is a group homomorphism.

$H \times H \xrightarrow{\mu^H} H \xhookrightarrow{i} G$ is smooth as a restriction of a smooth map ($\mu : G \times G \to G$) and

hence $\mu^H : H \times H \to H$ is ~~smooth~~ smooth by the universal property of initial submfld.

It remains to show uniqueness: Suppose $i: H \to G$ is a connected virtual Lie subgr. with Lie alg. $T_e H = \mathfrak{h} \subseteq \mathfrak{g}$.

Then $T_g H = T_e \lambda_g T_e H = E'_g$.

$\Longrightarrow$ H is an integral submfd. of E. Also, $e \in H$, implies $\underline{H \subseteq \mathcal{F}_e^E}$. Recall $\underline{\exp(\mathfrak{h})} \subseteq H \subseteq \mathcal{F}_e^E$.

Since $\exp(\mathfrak{h})$ generates $\mathcal{F}_e^E$ by Thm. 1.23, we conclude that $H = \mathcal{F}_e^E$.

$\square$.

Yesterday: G Lie group with Lie alg. $\mathfrak{g}$.

If $\mathfrak{h} \subseteq \mathfrak{g}$ is a subalgebra, then $\exists!$ connected virtual Lie subgroup $i : H \to G$ s.t. $i'(T_e H) = \mathfrak{h}$.

Thm 1.34 (Ado's Theorem)

Suppose $\mathfrak{g}$ is a finite-dim. Lie algebra. Then $\mathfrak{g}$ admits an injective representation $\varphi : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$ onto some finite-dim. vector space V.

In particular, $\mathfrak{g}$ is isomorphic to a subalgebra of $\mathfrak{gl}(V)$.

Proof. See literature.

Thm. 1.35 (Lie's 3rd Fundamental Theorem)
Let $\mathfrak{g}$ be a finite-dim. Lie algebra. Then $\exists$ a Lie group G with Lie algebra $\mathfrak{g}$.

Proof By Thm. 1.34, we can identify $\mathfrak{g}$ with a subalgebra of some $\mathfrak{gl}(V)$. Now ~~apply~~ Theorem 1.33 implies $\exists$ a virtual Lie subgroup $G \longrightarrow GL(V)$ with Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(V)$.

Remark But it is not true that any connected Lie group is isomorphic to a virtual Lie subgroup of some $GL(V)$.

1.4. Homogeneous spaces and Klein geometry

Lie groups arise as transformation groups of a space (vector space or mfd.) preserving some additional structure on that space $\longrightarrow$ seen Introduction.

Def. 1.36 G is a group, X a set.

$\ell : G \to Bij(X)$

A left action of G on X is a map $\ell : G \times X \longrightarrow X$ s.t.

$\ell(e,x) = x \quad \forall x \in X$ and $\ell(g, \ell(h,x)) = \ell(gh, x) \quad \forall g,h \in G \; \forall x \in X$.

For fixed $g \in G$ we write $\ell_g := \ell(g,-) : X \to X$

and for fixed $x \in X$ we write $\ell^x := \ell(-,x) : G \to X$.

Similarly, we have the notion of a right action of G on X.

It is given by a map $r : X \times G \to X$ s.t.

$r(x,e) = x$ and $r(r(x,g),h) = r(x,gh)$ $\forall x \in X, \forall g,h \in G$.

We set $r^g := r(-,g) : X \to X$ and $r_x := r(x,-) : G \to X$

for fixed $g \in G$ resp. $x \in X$.

<u>Notation</u>: We abbreviate $\ell(g,x) := g \cdot x = gx$ resp.

$r(x,g) := x \cdot g = xg$

Note that $g \mapsto \ell_g$ and $g \mapsto r^g$ define maps $\underline{G \to \mathrm{Bij}(X)}$

($\ell_{g^{-1}}$, $r^{g^{-1}}$ are the inverses of ℓ_g resp. r^g)

It is a group homomorphism for $g \mapsto \ell_g$ ($gh \mapsto \ell_{gh} = \ell_g \circ \ell_h$)

and a anti-group homomorphism for $g \mapsto r^g$ ($r^{gh} = r^h \circ r^g$).

<u>Remark</u> Given a right-action r, then $\ell_g := r^{g^{-1}}$ is a left action and conversely.

<u>Def 1.37</u> Given a left-action $\ell : G \times X \to X$, the **orbit of $x \in X$** is given by

$$G \cdot x = \mathrm{im}(\ell^x) = \{ gx : g \in G \} \subseteq X .$$

Similarly, one defines the orbit of a right-action.

<u>Prop. 1.38</u> ~~Given~~ Suppose $\ell: G \times X \to X$ is a left-action.

Then for points in X "being in the same orbit" defines an equivalence relation on X. ($x \sim y$ iff $\exists g \in G$ s.t. $x = gy$).

The set of equivalence classes $\underline{G \backslash X}$ is called the orbit space (of the action).

Similarly, for a right action and then we denote the orbit space by ~~G~~ X/G.

<u>Proof</u> If $x, y \in X$ s.t. $Gx \cap Gy \neq \emptyset$, then $\exists\, g, h \in G$ s.t. $gx = hy$. $\Longrightarrow x = \underline{g^{-1}h}\, y$

$\Longrightarrow x \in Gy$ and $Gx \subseteq Gy$ $\left(\tilde{g}x = \tilde{g}g^{-1}h\, y \in Gy \quad \forall \tilde{g} \in G\right)$.

By symmetry, also $Gy \subseteq Gx$ $\left(y = h^{-1}g\, x \in Gx\right)$.

Hence, $Gx = Gy$.

$\square$.

<u>Example</u>: G group, $H \subseteq G$ subgroup

$$\ell : H \times G \longrightarrow G$$
$$(h, g) \longmapsto \mu(h, g) = hg$$

$$r : G \times H \longrightarrow G$$
$$(g, h) \longmapsto gh.$$

define a left (resp. right) action of H on G.

$H\backslash G$ right coset space

G/H left coset space.

<u>Def. 1.39</u> $\varrho : G \times X \to X$ left action.

① ϱ is called transitive, if $Gx = X$ for a (hence, any) $x \in X$.

② For any $x \in X$, the stabilizer or isotropy group G_x of x is given by

$$G_x := \{ g \in G : gx = x \} \subseteq G .$$

Similarly, one defines the corresp. objects for right-actions.

Note $y \in Gx$, i.e. $y = gx$ for some $g \in G$,

$$G_y = G_{gx} = g^{-1} G_x g .$$

Moreover, note $\ell_x : G \to Gx$, $g \mapsto gx$ induces a bijection

$$G/G_x \xrightarrow{\sim} Gx .$$

Indeed, $g, h \in G$ s.t. $\ell_x(g) = gx = hx = \ell_x(h)$,

then $x = g^{-1}hx$, i.e. $g^{-1}h \in G_x$.

$\Longrightarrow$ $h \in gG_x$ and so $gG_x = hG_x$.

Def. 1.40 Suppose G is a group. Then a G-homogeneous space is a set X equipped with a transitive (left) action $\varrho : G \times X \to X$ of G.

In this case, for any point $x \in X$, we get a bijection

$$G/G_x \simeq X.$$

Under this identification, the left action of G on X becomes left multiplication by elements of G on G/G_x:

$$\varrho : G \times G/G_x \longrightarrow G/G_x$$
$$(g, \tilde{g} G_x) \longmapsto g\tilde{g} G_x.$$

Now, if G is a topolog. group (resp. a Lie group)
and X a topolog. space (resp. a smooth mfd.),
we can require an action to be continuous (resp. smooth),
i.e. we $\ell : G \times X \to X$ (resp. $r : X \times G \to X$) to
be continuous (resp. smooth).

<u>Ex.</u> G Lie group.

A representation of G is a smooth left action on a vector space
$X = V$, $\ell = \varphi : G \times V \to V$ s.t. $\varphi(g, \_) = \ell_g : V \to V$
is linear $\forall g \in G$.

Given a continuous left action of a topolog. group on a topolog. space, then $G\backslash X$ is naturally equipped with a topology:

$$\pi : X \longrightarrow G\backslash X$$

Equipp $G\backslash X$ with the quotient topology (= final topolog. w.r. to π), i.e. the finest topology s.t. π is continuous.

One has: $U \subseteq G\backslash X$ is open $\Longleftrightarrow$ $\pi^{-1}(U) \subseteq X$ is open.

For any topolog. space Y, ~~and~~ a map $f : G\backslash X \to Y$ is continuous $\Longleftrightarrow$ $f \circ \pi : X \to Y$ is continuous.

Topology on $G\backslash X$ might be „bad", even if G and X are „nice" topolog. spaces.

<u>Prop. 1.41</u> G topolog. group, $H \subseteq G$ a topolog. subgroup. and $\pi : G \to \underline{G/H}$ the natural continuous projection.

① $\ell_g : G/H \to G/H \quad \ell_g(g'H) = gg'H \quad \forall g \in G.$

is continuous $\quad$ ($G \times \underline{G/H} \to G/H$ continuous left action).

② G/H is Hausdorff $\Longleftrightarrow$ H is closed (topolog)

Proof

(1) $U \subseteq G/H$ open subset, then we need to show that

$(\ell_g)^{-1}(U)$ is open $\forall g \in G$.

Set $V := \{(g, g') \in G \times G : g \cdot g' \in \pi^{-1}(U)\}$

$= \mu^{-1}(\pi^{-1}(U)) \subseteq G \times G$.

is open in $G \times G$, since π and μ are continuous.

$\pi' : G \times G \to G \times G/H$ is continuous and open

$\shortparallel$

$Id_G \times \pi$

since $\pi : G \to G/H$ is open.

$\Longrightarrow \pi'(V) = \{(g, g'H) : gg'H \in U\} = (\ell_g)^{-1}(U)$

is open.

② '$\Rightarrow$' G/H is Hausdorff $\Rightarrow$ points are closed.

Since $\pi : G \to G/H$ is continuous, $H = \pi^{-1}(eH)$ is closed.

'$\Leftarrow$' Assume $H \subseteq G$ is closed.

$\psi : G \times G \to G$ $\quad \psi(g, \tilde{g}) = g^{-1}\tilde{g}$ is continuous.

$\psi^{-1}(H) = \{(g, \tilde{g}) : gH = \tilde{g}H\} \subseteq G \times G$

is closed.

For any pair $(g, \tilde{g}) \in \underline{G \times G \setminus \psi^{-1}(H)}$, $\exists$ open neighbh. U and $\tilde{U}$ of g resp. $\tilde{g}$ in G.

s.t. $U \times \tilde{U}$ is an open neigh. of $(g, \tilde{g})$ in $G \times G$ not intersecting $\varphi^{-1}(H)$.

$\Rightarrow$ $\pi(U)$ and $\pi(\tilde{U})$ are open neighbh. of $\underline{gH \neq \tilde{g}H}$ respectively that don't intersect (by construction).

$\square$.

Note that, if G (top. group) is acting on a topological Hausdorff space X, then $G_x \subseteq G$ is a closed subgroup and hence G/G_x is Hausdorff. ($G/G_x \xrightarrow{\sim} G_x$ continuous bijection, but not a homeomorphism in general).

Thm. 1.42 Suppose G is a Lie group and $H \leq G$ a closed subgroup (hence a Lie subgroup by Thm. 1.29).

Then the homogeneous space G/H admits a unique structure of a smooth mfd. s.t. $\pi : G \to G/H$ is a smooth submersion (i.e. $T_g\pi : T_g G \to T_{gH} G/H$ is surj. $\forall g \in G$).

In particular, $\dim(G/H) = \dim(G) - \dim(H)$.

Moreover, $\ell : G \times G/H \to G/H$, $\ell(g, g'H) = gg'H$, is a smooth left-action of G on G/H.