- M mfd. $\rightsquigarrow$ $\mathcal{F}(M) \to M$ frame bundle $GL(n,\mathbb{R})$-princ. bundle

$\mathcal{F}_x(M) = \{\text{linear isomorph. } \mathbb{R}^n \to T_xM\} \simeq \{\text{all basis of } \underline{T_xM}\}$

$$\begin{array}{c} \mathcal{F}(M) \\ \downarrow \uparrow s \\ M \end{array}$$

<u>Sections</u>

$s(x)$ is a basis of T_xM for $x \in M$.

- $V \to M$ vector bundle with standard fibre $\mathbb{V}$ $\rightsquigarrow$ $\mathcal{F}_x(V) = \{\text{linear isom. } \underline{\mathbb{V} \to} \underline{V_x}\}$

$$\begin{array}{c} \mathcal{F}(V) \\ \downarrow \\ M \end{array}$$

$GL(\mathbb{V})$-principal bundle.

Def. 2.15 Suppose M mfd. of dim. n and $G \subseteq GL(n,\mathbb{R})$ a closed subgroup.

Then a (first-order) G-structure on M (with structure group G) is a reduction of structure group of $\mathcal{F}(M) \to M$ to $G \subseteq GL(n,\mathbb{R})$.

$$\begin{tikzcd} P \arrow[r, hook] \arrow[dr] & \mathcal{F}(M) \arrow[d] \\ & M \end{tikzcd} \qquad \text{w.r.t. } i : G \hookrightarrow GL(n,\mathbb{R}).$$

G-principal bundle

Many geometric structures on mfds can be equiv. described as G-structures:

Examples

(1) Riemannian mfds.

(M, g) Riem. mfd. $\quad g_x$ is a positive definite inner product on T_xM

$(\mathbb{R}^n, \langle\,,\,\rangle)$

standard inner product.

$\langle v, w\rangle = g_x(u(v), u(w))$

$O_x(M) := \{u : \mathbb{R}^n \to T_xM$ linear isomorphism. that are orthogonal w.r. to $\langle\,,\,\rangle$ and $g_x\} \subseteq F_x(M)\} \simeq \{$ortho. normal basis w.r.to $g_x\}$

$$\begin{tikzcd} \underline{\mathcal{O}(M) := \bigsqcup_{x\in M} \mathcal{O}_x(M)} \arrow[r, hook] \arrow[dr, "q"'] & \mathcal{F}(M) = \bigsqcup_{x\in M} \mathcal{F}_x(M) \arrow[d] \\ & M \end{tikzcd}$$

$$O(n) \hookrightarrow GL(n, \mathbb{R})$$

$\mathcal{O}(M) \to M$ is a principal bundle with structure $O(n)$, called the orthonormal frame bundle of (M, g).

Gram–Schmidt procedure can be used to construct local orthonormal frames of TM, i.e. we can find a cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ of M and smooth section $s_\alpha : \underline{U_\alpha} \to \mathcal{F}(M)$ with values in $\mathcal{O}(M)$.

Then $\phi_\alpha^{-1} : U_\alpha \times O(n) \longrightarrow q^{-1}(U_\alpha)$ $\qquad$ $\underline{S_\alpha(x) :} \ \mathbb{R}^n \to T_x M$

$$\phi_\alpha^{-1}(x, A) = S_\alpha(x) \circ A : \mathbb{R}^n \to T_x M \in \mathcal{O}_x(M)$$

are bijections and $\phi_\beta \circ \phi_\alpha^{-1}(x, A) = (x, \psi_{\beta\alpha}(x) \circ A)$,

where $\psi_{\beta\alpha}(x) \in O(n)$ characterized by $S_\alpha(x) = S_\beta(x)\, \psi_{\beta\alpha}(x)$

$\left(\psi_{\beta\alpha}(x) = \cancel{S_\alpha(x)} \; S_\beta(x)^{-1} \circ S_\alpha(x) : \mathbb{R}^n \to \mathbb{R}^n \right)$

By Prop. 2.4, $\mathcal{O}(M) \to M$ is a princ. $O(n)$-bundle

and the inclusion $\mathcal{O}(M) \hookrightarrow \mathcal{F}(M)$ is a reduction of str. gr.,

i.e. a $O(n)$-structure.

Conversely, let $Q \hookrightarrow \mathcal{F}(M)$ be a reduction to $O(n)$.

$$\begin{tikzcd} Q \arrow[r, hook] \arrow[dr] & \mathcal{F}(M) \arrow[d] \\ & M \end{tikzcd}$$

Fix $x \in M$, $u \in Q_x \subseteq \mathcal{F}_x(M)$ $\qquad u : \mathbb{R}^n \to T_x M$

$$Q_x = u \cdot \underline{O(n)}$$

$$g_x(\xi_x, \eta_x) := \langle u^{-1}(\xi_x), u^{-1}(\eta_x) \rangle .$$

is a positive definite inner product on $T_x M$ that does not on $u \in Q_x$ (u', $u' = u \circ A$

$$\longrightarrow \langle u'^{-1}(\xi_x), (u')^{-1}(\eta_x) \rangle = \langle A^{-1} \circ u^{-1}(\xi_x), A^{-1} \circ u^{-1}(\eta_x) \rangle$$
$$\underset{A \in O(n)}{=} \langle u^{-1}(\xi_x), u^{-1}(\eta_x) \rangle$$

Also, g_x depends smoothly on x.

Choose a local section $s: U \to Q$, then $\forall y \in U$ $g_y = \langle s(y)^{-1}(\_), s(y)^{-1}(\_)\rangle$ depends smoothly on y.

Hence, Riemannian metrics on M are the same as first order G-struct. with structure group $O(n)$ M.

Similarly, Pseudo-Riemannian metrics of signature (p,q) on M

$$\rightleftharpoons O(p,q)\text{-structures on } M.$$

(2) Reductions of $\mathcal{F}(M)$ to $GL_+(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) : \det(A) > 0\}$

$\rightleftharpoons$ orientations on M.

(3) $H \subseteq GL(n,\mathbb{R})$ is the stabilizer of $\mathbb{R}^k \subseteq \mathbb{R}^n$

$\left(H = \{ A \in GL(n,\mathbb{R}) \; ; \; A\,\mathbb{R}^k = \mathbb{R}^k \} \right)$

Distributions of rank k $\underline{E \subseteq TM}$ $\qquad \boxed{TM = E \oplus E'}$

$\rightleftharpoons$

H-structures on M.

(4) Volume forms on M $\rightleftharpoons$ $SL(n,\mathbb{R})$-structures on M.

(5) M mfd, $\dim(M) = 2n$.

An almost symplectic structure on M is a non-degenerate 2-form $\underline{\omega} \in \Gamma(\Lambda^2 T^*M) = \Omega^2(M)$

Almost symplectic structures on M $\rightleftharpoons$ $Sp(2n,\mathbb{R})$-structures on M.

(6) M mfd, $\dim(M) = 2n$

An almost complex structure on M is vector bundle map $J: TM \to TM$ s.t. $J^2 = -\mathrm{Id}$.

Almost complex structures on M $\rightleftharpoons$ $GL(n,\mathbb{C})$-structures on M

$$GL(n,\mathbb{C}) \subseteq GL(2n,\mathbb{R})$$
$$\| $$
$$\{ A \in GL(2n,\mathbb{R}) : A \circ I = I \circ A \}$$

I is standard complex. str. on $\mathbb{R}^{2n}$

$$\Big((T_xM, J_x) \simeq (\mathbb{R}^{2n}, I) \Big)_x .$$

2.5 Some Constructions with fiber bundles

Suppose M and N are (smooth) mflds, $p : E \to N$ fiberbundle,

$f : M \to N$ C^∞-map.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

$$f^*E := \{(x,u) \in M \times E : f(x) = p(u)\} \subseteq M \times E \text{ closed subset.}$$

Restricting the projections $\underline{M \times E \to E}$ and $M \times E \to M$ to f^*E gives rise to maps $p^*f : f^*E \to E$ and $f^*p : f^*E \to M$

Commuting diagram:

$$\begin{tikzcd} \underline{f^*E} \arrow[r, "p^*f"] \arrow[d, "f^*p"'] & E \arrow[d, "p"] \\ \underline{M} \arrow[r, "f"] & N \end{tikzcd}$$

For $x \in M$,

$$(f^*E)_x = E_{f(x)}$$

If $\phi : p^{-1}(U) \to U \times F$ is fiber bundle chart of $p: E \to N$,

then $f^{-1}(U)$ is open in M and

$$\tilde{\phi} : (f^*p)^{-1}(\underline{f^{-1}(U)}) \longrightarrow \underline{f^{-1}(U)} \times \underline{F}$$

$$\tilde{\phi}(x, u) = (x, pr_2(\phi(u)))$$

$f(x) \in U$, $u \in E_{f(x)}$

is a bijection.

Transition maps :

$$\tilde{\Phi}_\beta \circ \tilde{\Phi}_\alpha^{-1}(x,y) \longmapsto (x, \phi_{\beta\alpha}(f(x), y))$$

are smooth. By Prop. 2.4, $f^*p : f^*E \to M$ is a fiber bundle and $p^*f : f^*E \to E$ is a morphism of fiber bundles covering $f: M \to N$.

Def. 2.16 $f: M \to N$ C^∞-map between mflds., $p: E \to N$ a fiber bundle. Then the fiber bundle $f^*p : f^*E \to N$ is called the pullback of p along f.

Note that, if E is a vector bundle (resp. principal bundle) so is f^*E. For any local section s of $E \to N$, $f^*s := s \circ f$ is a section of f^*E.

<u>Universal property</u>: For any fiber bundle $\tilde{E} \xrightarrow{q} M$ and a morphism $\hat{q} : \tilde{E} \to E$ of fiber bundles covering f, $\exists!$ morphism Q of fiber bundles $\hat{E} \to f^*E$ making

$$\begin{tikzcd}
\tilde{E} \arrow[dr, dashed, "Q"] \arrow[drr, bend left, "\hat{q}"] \arrow[ddr, bend right, "q"'] & & \\
& f^*E \arrow[r, "p^*f"] \arrow[d, "f^*p"'] & E \arrow[d, "p"] \\
& M \arrow[r, "f"] & N
\end{tikzcd}$$

commute.

For $u \in \hat{E}$ with $q(u) = x \in M$, $\hat{q}(u) \in E_{f(x)}$,

$$Q(u) := (q(u), \hat{q}(u)) \in f^*E.$$

Def. 2.17 $p : E \to M$, $\tilde{p} : \tilde{E} \to M$ fiber bundles over M with standard fibers F and $\tilde{F}$ resp. Then their fiber product is given by

$$E \times_M \tilde{E} \subset E \times \tilde{E}$$

$$\| $$

$$\{ (u, \tilde{u}) \in E \times \tilde{E} : p(u) = \tilde{p}(\tilde{u}) \}$$

$$\left(E \times_M \tilde{E} = p^* \tilde{E} = \tilde{p}^* E \right)$$

We have a natural projection $q : E \times_M \tilde{E} \longrightarrow M$

$$(u, \tilde{u}) \longmapsto p(u) = \tilde{p}(\tilde{u})$$

and $q^{-1}(x) = E_x \times \tilde{E}_x \quad \forall x \in M$.

This is a fiber bundle over M with standard fiber $F \times \tilde{F}$.

$\phi : p^{-1}(U) \to U \times F \quad , \quad \tilde{\phi} : \tilde{p}^{-1}(U) \to U \times \tilde{F}$

charts for E and $\tilde{E}$

$\Rightarrow \quad \psi : q^{-1}(U) \to U \times F \times \tilde{F}$

$$(u, \tilde{u}) \longmapsto \big(p(u) = \tilde{p}(\tilde{u}), \, pr_2(\phi(u)), \, pr_2(\tilde{\phi}(\tilde{u}))\big)$$

is a bijection.

Transition maps for ψ's are smooth, and Prop. 2.4 implies $q : E \times_M \tilde{E} \to M$ is a fiber bundle over M.

Moreover,
$$\begin{tikzcd} E \times_M \widetilde{E} \arrow[r, "pr_1"] \arrow[dr, "q"'] & E \arrow[d, "r"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} E \times_M \widetilde{E} \arrow[r, "pr_2"] \arrow[dr, "q"'] & \widetilde{E} \arrow[d, "\widetilde{r}"] \\ & M \end{tikzcd}$$

define fiber bundle morphisms (over id_M).

<u>Universal property</u>: Suppose $S \to N$ is a fiber bundle and $\Phi: S \to E$, $\widetilde{\Phi}: S \to \widetilde{E}$ morphism of fiber bundles covering the same map $f: M \to N$ then $\exists !$ morphism $(\Phi, \widetilde{\Phi}): S \to E \times_M \widetilde{E}$ with base map f s.t. $pr_1 \circ (\Phi, \widetilde{\Phi}) = \Phi$ and $pr_2 \circ (\Phi, \widetilde{\Phi}) = \widetilde{\Phi}$.

Moreover, the fibered product of princ. bundles over M with structure groups G and $\tilde{G}$ resp. is a principal $G \times \tilde{G}$-bundle over M.

For vector bundles V and $\tilde{V}$ over M, $V \times_M \tilde{V} = V \oplus \tilde{V} \to M$ (i.e. fibered product equals direct sum).

For $s \in \Gamma(E)$, $\tilde{s} \in \Gamma(\tilde{E})$, $s \times \tilde{s} \in \Gamma(E \times_M \tilde{E})$.

$\rightsquigarrow$ Bijection $\Gamma(E) \times \Gamma(\tilde{E}) \simeq \Gamma(E \times_M \tilde{E})$.

Prop. 2.18 Suppose $P \xrightarrow{p} M$ principal G-bundle. Then

$\exists$ a smooth map $\tau : P \times_M P \longrightarrow G$ s.t.

for $u, v \in P$ with $p(u) = p(v)$ one has $\underline{v = u \cdot \tau(u,v)}$.

Proof. For $u, v \in P_x$ $\exists!$ $g \in G$ s.t. $v = u \cdot g$.

Set $\tau(u,v) := g$.

Let $\phi : p^{-1}(U) \to U \times G$ ~~princ~~ principal bundle chart

for $p : P \to M$, then

$$(u,v) \longmapsto \Big(\underset{\substack{\shortparallel \\ p(u)=p(v)}}{x} ,\ pr_2(\phi(u)),\ pr_2(\phi(v)) \Big)$$

is one for $P \times_M P$.

By defin., $T(u,v) = pr_2(\phi(u))^{-1} p_2(\phi(v))$

and hence smoothness follows.

<u>Cor. 2.19</u> $p : P \to M$ principal G-bundle

p is isomorphic to $M \times G \to M \iff p$ admits a global section.

<u>Proof.</u> '$\Longrightarrow$' ✓

'$\Longleftarrow$' $s : M \to P$ global section of $p : P \to M$,

then $M \times G \longrightarrow P$ is a smooth bijection.
$(x,g) \longmapsto s(x)\cdot g$

with inverse $u \longmapsto (p(u), \tau(s(p(u)), u))$ which is smooth too.

$(x, g) \longmapsto s(x)g$ defines a trivialization

$$\begin{tikzcd} M \times G \arrow[rr, "\sim"] \arrow[dr] & & P \arrow[dl] \\ & M & \end{tikzcd}$$