2.6 Associated bundles

Suppose $p: P \to M$ is a principal G-bundle and let $G \times F \to F$ be a smooth left-action of G on a mfd. F.

$$P \times F \times G \longrightarrow P \times F$$
$$(u, f) \cdot g := (ug, g^{-1} f)$$

defines a smooth right-action of G on $P \times F$.

Denote by $P \times_G F := P \times F / G$ orbit space of this action.

We write $[u, f]$ for the orbit / equiv. class of $(u, f) \in P \times F$.

There are two natural maps:

- $q : P \times F \longrightarrow P \times_G F$

  $(u, f) \longmapsto [u, f]$

- $\pi : P \times_G F \longrightarrow M$

  $[u, f] \longmapsto p(u)$

(well-defined, $(ug, g^{-1} f)$

$p(ug) = p(u)$)

Prop. 2.20 Suppose $p: P \to M$ is a principal G-bundle and $G \times F \to F$ a smooth left-action of G on a mfd. F.

① $\pi : P \times_G F \longrightarrow M$ is a (smooth) fiber bundle with stand. fiber F and structure group G, called the associated bundle to $p: P \to M$ corresp. to $G \times F \to F$.

If $F = W$ is a vector space and $G \times W \to W$ a represent., then $V := P \times_G W \longrightarrow M$ is a vector bundle.

② The projection $q: P \times F \longrightarrow P \times_G F$ is a principal bundle with structure G.

③ There is a smooth map $\tau_F : \underline{P \times_H (P \times_G F)} \to F$

characterized by the property that:

For $z \in P \times_G F$, $u \in P$ s.t. $\pi(z) = p(u)$ $\quad$ $\left((u,z) \in P \times_H (P \times_G F)\right)$

one has $\quad z = q(u, \tau_F(u,z))$.

In particular, $\tau_F(u \cdot g, z) = g^{-1} \cdot \tau_F(u,z)$.

<u>Proof.</u>

① Choose a principal G-bundle atlas for $p : P \to M$

$\{(U_\alpha, \phi_\alpha) \mid \alpha \in I\}$. $\quad \phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \underline{G}$ $\qquad G \times F \to F$

Now define: $\underline{\Phi_\alpha} : \pi^{-1}(U_\alpha) \longrightarrow \underline{U_\alpha \times F}$

$[u,f] \longmapsto (\underline{p(u)}, \; pr_2(\phi_\alpha(u)) \cdot f)$

- $\pi([u,f]) = p(u)$ (by definition of π).

$[u,f] = [u',f'] \Longrightarrow \exists g \in G$ s.t. $u' = u \cdot g$ and $f' = g^{-1} f$

$$\Longrightarrow pr_2(\phi_\alpha(u')) \cdot f' = \underbrace{pr_2(\phi_\alpha(ug))}_{= pr_2(\phi_\alpha(u)) \cdot g} \cdot g^{-1} f = pr_2(\phi_\alpha(u)) \cdot f$$

cf. Lemma 2.10.

$\Longrightarrow \overline{\Phi_\alpha}$ is well-defined and $\pi|_{\pi^{-1}(U_\alpha)} = pr_1 \circ \overline{\Phi_\alpha}$.

- $\overline{\Phi_\alpha}$ is bijective: Given $(x,f) \in U_\alpha \times F$, then

$$\overline{\Phi_\alpha}([u,f]) = (x,f) \quad \text{for } u := \phi_\alpha^{-1}(x,e)$$

so $\overline{\Phi_\alpha}$ is surjective.

Assume $\underline{\Phi_\alpha}([u,f]) = \Phi_\alpha([u',f'])$. Then $p(u) = p(u')$,

so $\exists g \in G$ s.t. $\underline{u' = u \cdot g}$. Hence, $pr_2(\phi_\alpha(u')) = pr_2(\phi_\alpha(u)) \cdot g$

and $pr_2(\phi_\alpha(u')) \cdot f' = \underline{pr_2(\phi_\alpha(u))} \cdot g f' = \underline{pr_2(\phi_\alpha(u))} \cdot f$

by assumption

$\Rightarrow g \cdot f' = f \iff \underline{f' = g^{-1} f}$

$\Rightarrow [u,f] = [u',f']$.

Hence, Φ_α is bijective.

· Transition maps : $x \in U_{\alpha\beta}$, $f \in F$

$$\overline{\Phi}_\alpha \circ \overline{\Phi}_\beta^{-1} : (x, f) \longmapsto [\overline{\Phi}_\beta^{-1}(x,e), f] \longmapsto (x, \underbrace{pr_2(\Phi_\alpha \circ \Phi_\beta^{-1}(x,e))}_{\psi_{\alpha\beta}(x)} \cdot f)$$

(cocycle)

where $\psi_{\alpha\beta} : U_{\alpha\beta} \to G$ are transition maps of the principal bundle atlas $\{(U_\alpha, \Phi_\alpha)\}$. Therefore, 1 holds again from Prop. 2.4.

2 Take a principal bundle chart (U_α, Φ_α) for $p : P \to M$ and let $(U_\alpha, \overline{\Phi}_\alpha)$ be the corresp. chart for $\pi : P \times_G F \to M$ as in 1. Then $q : \underline{P \times F} \to P \times_G F$ is smooth, since

$$\Phi_\alpha\left(q\left(\phi_\alpha^{-1}(x,g),f\right)\right) = (x, g^{-1}f)$$

is smooth.

For the open subset $\pi^{-1}(U_\alpha) \subseteq P\times_G F$, one has

$$q^{-1}\left(\pi^{-1}(U_\alpha)\right) = p^{-1}(U_\alpha)\times F$$

Define $\psi_\alpha : p^{-1}(U_\alpha)\times F \longrightarrow \pi^{-1}(U_\alpha)\times G$ by

$$\psi_\alpha(u,f) = \left([u,f],\, pr_2(\phi_\alpha(u))\right).$$

Then one verifies directly that $\{(U_\alpha,\psi_\alpha) : \alpha\in I\}$ is a principal fibre bundle atlas for $q : P\times F \longrightarrow P\times_G F$.

③ $(u, z) = (u, [u', f]) \in P \times_M (P \times_G F)$

$p(u) = \pi([u', f]) = p(u')$

$\Rightarrow \exists\, g \in G$ s.t. $u' = u \cdot g$ and hence $[u', f] = [ug, f]$

$= [u, g^{-1} f]$

If $[u, f] = [u, f']$, then $f = f'$, so τ_F is well-defined.

Remains to check it is smooth. <u>Exercise</u>.

$\square$.

Suppose $p: P \to M$ is a principal G-bundle, $G \times F \to F$ left-action.

A smooth map $f: P \longrightarrow F$ is G-equivariant, if $f(u \cdot g) = g^{-1} \cdot f(u)$ $\forall u \in P$, $\forall g \in G$. We write $C^\infty(P, F)^G = \{ f \in C^\infty(P, F) :$ f is G-equiv.$\}$.

<u>Cor. 2.21</u> $p: P \to M$ princ. G-bundle, $G \times F \to F$ left-action of G. Then there is a natural bijection:

$$\underline{\Gamma(P \times_G F)} \cong C^\infty(P, F)^G.$$

Proof $s \in \Gamma(P \times_G F) \rightsquigarrow f_s : P \to F$

$f_s(u) := \tau_F(u, \underline{s(p(u))})$ $\quad (s(p(u)) = [u, \underline{f_s(u)}])$

smooth map

and $f_s(ug) = \tau_F(u \cdot g, s(p(u \cdot g))) = \tau_F(u \cdot g, s(p(u)))$

$s(p(u)) = s(p(u \cdot g)) = [ug, f_s(ug)] = [\underline{u}, \underline{g^{-1} \cdot f_s(u \cdot g)}]$

$\parallel$

$[\underline{u}, \underline{f_s(u)}] \implies f_s(u \cdot g) = g^{-1} \cdot f_s(u).$

Conversely, $f : P \to F$ C^∞-map + G-equiv.,

$s_f \in \Gamma(\underline{P \times_G F})$

Smoothness of s_f:

$s_f(x) = q(\sigma(x), f(\sigma(x)))$

$= [\sigma(x), f(\sigma(x))]$

for a local section σ of P.

$\underline{s_f(x)} = [u, f(u)]$ $\quad u \in P_x$.

well-defined, since $u' \in P_x$ is of the form $u' = u.g$ for $g \in G$ and $\underline{[u', f(u')]} = [u.g, f(u.g)]$

$= [u.g, g^{-1} f(u)] = \underline{[u, f(u)]}$

$s \mapsto f_s$

$s_f \leftarrow f$

are inverse to each other.

Examples

① M mfd. of dim. n. $\mathcal{F}(M) \longrightarrow M$ frame bundle (principal $GL(n,\mathbb{R})$-bundle).

$GL(n,\mathbb{R}) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ standard repres.

$$\mathcal{F}(M) \times_{GL(n,\mathbb{R})} \mathbb{R}^n \cong TM.$$

$\mathcal{F}(M) \times \mathbb{R}^n \longrightarrow TM$ is a surjection.

$$(u, y) \longmapsto u(y) \in T_xM \qquad (uA, A^{-1}y) \mapsto u \circ A A^{-1} y = u(y).$$

$u : \mathbb{R}^n \to T_xM$

that factor to a bijection $\tilde{F}(M) \times_{GL(n,\mathbb{R})} \mathbb{R}^n \simeq TM$.

$$\begin{tikzcd} \tilde{F}(M) \times_{GL(n,\mathbb{R})} \mathbb{R}^n \arrow[r, "\simeq"] \arrow[dr] & TM \arrow[d] \\ & M \end{tikzcd}$$

which is an isomorphism of vector bundles.

$\Longrightarrow$ $\underline{\tilde{F}(M) \times_{GL(n,\mathbb{R})} \mathbb{R}^{n*} \simeq T^*M}$

$\tilde{F}(M) \times_{GL(n,\mathbb{R})} \Lambda^k \mathbb{R}^{n*} \simeq \Lambda^k T^*M$

$\tilde{F}(M) \times_{GL(n,\mathbb{R})} \otimes^p \mathbb{R}^n \otimes \otimes^q \mathbb{R}^{n*} \simeq \otimes^p TM \otimes \otimes^q T^*M$,

Similarly, $V \to M$ with standard fiber $\mathbb{V}$, then there is a natural isomorphism of vector bundles

$\tilde{F}(V) \times_{GL(\mathbb{V})} \mathbb{V} \simeq V$

(2) $G \subseteq GL(n, \mathbb{R})$ closed subgr., M mfld. of dim. n

equipped with a G-structure:

$$\begin{tikzcd} P \arrow[r, hook, "\varphi"] \arrow[dr] & F(M) \arrow[d] \\ & M \end{tikzcd}$$

$G \times \mathbb{R}^n \to \mathbb{R}^n$ restriction of standard repres. to G.
of $GL(n, \mathbb{R})$

$P \times_G \mathbb{R}^n \simeq TM$ $\qquad \varphi(u) : \mathbb{R}^n \to T_x M \qquad u \in P_x$.

$[u, y] \mapsto \varphi(u)(y)$

$(ug, g^{-1}y) \mapsto \underbrace{\varphi(u \cdot g)}(g^{-1}y)$

$\varphi(u) g g^{-1} y = \varphi(u)(y)$

Question: Can any fiber bundle with structure group G (in particular, any vector bundle) be realized as an associated bundle to a principal G-bundle?

Prop. 2.22 Suppose $\pi : E \to M$ is a fiber bundle with standard F and structure group G acting effectively on F.
Then there is a unique (up to isomorphism) G-principal bundle $p : P \to M$ s.t. $E \cong P \times_G F$ as bundles with structure group G.

<u>Proof</u> (Sketch)

$\psi_{\alpha\beta} : U_{\alpha\beta} \longrightarrow G$.

Let $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in I}$ be a G-atlas for E and $\{\psi_{\alpha\beta}\}_{(\alpha,\beta) \in I \times I}$ the corresp. cocycles. $(\Phi_\alpha \circ \Phi_\beta^{-1}(x, f) = (x, \psi_{\alpha\beta}(x) \cdot f)$.

By Prop. 2.6, we can use the cocycles $\{\psi_{\alpha\beta}\}$ to construct a principal G-bundle $p : P \longrightarrow M$ $\left(P = \bigsqcup_{\alpha \in I} U_\alpha \times G \,/\, \sim\right)$ with transition fcts $(x, g) \longmapsto (x, \psi_{\alpha\beta}(x) \cdot g)$ using Prop. 2.6

Taking the induced atlas $\{(U_\alpha, \overline{\Phi}_\alpha)\}$ for $P \times_G F$, the transition fcts are given by $(x, f) \longmapsto (x, \psi_{\alpha\beta}(x) \cdot f)$.

and $\underline{H(\Phi_\alpha^{-1}(x,f)) = \Phi_\alpha^{-1}(x,f)}$ fits together to

define an isomorphism $H : \underline{P \times_G F} \xrightarrow{\sim} \underline{E}$.

of fiber bundles

with structure group G.

<u>Uniqueness</u> : Suppose $\tilde{P} \to M$ G-principal bundle with

cocycles $\{\tilde{\varphi}_{\alpha\beta}\}$ and we have an isomorphism $H : \underline{P \times_G F} \simeq \tilde{P} \times_G F$

of bundles with structure group G.

Then for any $\alpha \in I$, $\tilde{\Phi}_\alpha \circ H\big|_{\pi^{-1}(U_\alpha)} : \pi^{-1}(U_\alpha) \to U_\alpha \times F$

must be G-compatible to chart $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$

$\Rightarrow$ $\exists$ a smooth map $f_\alpha : U_\alpha \to G$ s.t. $\overline{\Phi}_\alpha(H(\phi_\alpha^{-1}(x,f))$
$= (x, f_\alpha(x)\cdot f)$.

$\rightsquigarrow$ for any $x \in U_{\alpha\beta}$, $f \in F$ we get for f_α, f_β s.t.

$$\underline{f_\beta(x)}\cdot \underline{f} = \underline{\tilde{\psi}_{\beta\alpha}(x) f_\alpha(x) \psi_{\alpha\beta}(x)}\cdot f$$

By effectivity of $G \times F \to F$, $\tilde{\psi}_{\beta\alpha}(x)\, f_\beta(x) = f_\alpha(x)\psi_{\alpha\beta}$

(i.e. the cocycles are cohomologous) and so $P \simeq \tilde{P}$.

$\square$.

Functorial properties of associated bundles :