Yesterday:

We define notion of a Lie group (G, μ, ν, e)

- $\lambda_g : G \longrightarrow G$ $\qquad \lambda_g(h) = \mu(g,h)$

  $\rho^g : G \to G$

Formulas: $T\mu$, $T\nu$

Recall from Global Analysis: $f : M \longrightarrow M$ is a diffeom. of a smooth mfd. M. Pull back: $f^* : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ $\left(f^*\xi = (Tf)^{-1} \circ \xi \circ f \right)$ is linear and $f^*[\xi, \eta] = [f^*\xi, f^*\eta]$ $\forall \xi, \eta \in \mathfrak{X}(M)$.

<u>Def. 1.6</u> Suppose G is a Lie group. Then a vector field $\xi \in \mathfrak{X}(G)$ is called left- (resp. right-) invariant, if

$$\lambda_g^* \xi = \xi \quad (\text{resp. } (\rho^g)^* \xi = \xi) \quad \forall g \in G.$$

Denote by $\mathfrak{X}_L(G)$ (resp. $\mathfrak{X}_R(G)$) the subset of $\mathfrak{X}(G)$ of left (resp. right) invariant vectorfield. By linearity of the pullback, $\mathfrak{X}_L(G)$ and $\mathfrak{X}_R(G)$ are subspaces of $\mathfrak{X}(G)$. They are even subalgebras of the infinite dimensional Lie algebra $(\mathfrak{X}(G), [\,,\,])$ by compatibility of $[\,,\,]$ with the pullback.

Lie bracket of vector fields

Prop. 1.7 Suppose G is a Lie group and set $\mathfrak{g} := T_e G$.

① For any $X \in \mathfrak{g}$,

$$L_X(g) := T_e \lambda_g X \in T_g G \quad (\text{resp. } R_X(g) := T_e \rho_g X \in T_g G)$$

is a left- (resp. right-) invariant vector field on G.

② The maps $G \times \mathfrak{g} \to TG$ defined by $(g, X) \mapsto L_X(g)$ and $(g, X) \mapsto R_X(g)$ are diffeomorphism.

③ The map $X \mapsto L_X$ (resp. $X \mapsto R_X$) defines a linear isomorphism with inverse $\xi \mapsto \xi(e)$ between

$\mathfrak{g}$ and $\mathfrak{X}_L(G)$ (resp. $\mathfrak{X}_R(G)$).

<u>Proof.</u>

① By ② L_X and R_X are smooth vector fields. Let us check that L_X is left-invariant:

$$(\lambda_g^* L_X)(h) = (T\lambda_g)^{-1} \circ L_X(gh) = (T_{gh}\lambda_{g^{-1}})\, \overbrace{T_e\lambda_{gh}}^{T_e(\lambda_g\circ\lambda_h)} X$$

$$= T_e(\underbrace{\lambda_{g^{-1}}\circ\lambda_g}_{\mathrm{Id}}\circ\lambda_h)X = L_X(h) \qquad \forall h \in G$$

$\lambda_g^* L_X = L_X \quad \forall g \in G$. Similarly, one shows that R_X is right-invariant.

(2) Define the map $F : G \times \mathfrak{g} \to TG \times TG$ given by $F(g, X) := (\underbrace{0_g}_{\in T_gG}, X)$.

It is smooth and so is $T\mu \circ F : G \times \mathfrak{g} \to TG$ as composition of smooth maps.

By Lemma 1.5, $\underline{T\mu \circ F : (g, X) \mapsto L_X(g)}$

To show $T\mu \circ F$ is a diffeomorphism we construct a smooth inverse of $T\mu \circ F$. Define $\tilde{F} : TG \to TG \times TG$ by $\tilde{F}(\xi_g) := (0_{g^{-1}}, \xi_g) \in T_{g^{-1}}G \times T_gG$.

It is smooth (since inversion is smooth) and so is $T\mu \circ \tilde{F}$, which by Lemma 1.5 is given by

$$T\mu \circ \tilde{F}: \xi_g \longmapsto T_g \lambda_{g^{-1}} \xi_g \in T_e G = \mathfrak{g}.$$

$\Longrightarrow$ The map $\underset{\in T_g G}{\xi_g} \longmapsto (g, T_g \lambda_{g^{-1}} \xi_g) \in G \times \mathfrak{g}$

$$TG \longrightarrow G \times \mathfrak{g}$$

is smooth and it is an inverse to $(g, X) \mapsto L_X(g)$

$$G \times \mathfrak{g} \longrightarrow TG.$$

Similarly, one proves the statement for $(g, X) \mapsto R_X(g)$.

③ By ①, $X \longmapsto L_X$ defines a linear map

$$\mathfrak{g} \longrightarrow \mathfrak{X}_L(G) \qquad L_X(g) = T_e\lambda_g X$$

$$\Big(\underbrace{X \longmapsto L_X \longrightarrow L_X(e)}_{\mathrm{Id}_{\mathfrak{g}}} = X \Big).$$

If $\xi \in \mathfrak{X}_L(G)$, then $\xi(g) = (\lambda_{g^{-1}})^* \xi(g)$

$$= T_e\lambda_g\, \xi(e) = L_{\xi(e)}(g) \quad \forall g \in G.$$

Similarly, for $X \longmapsto R_X$. $\square$

Def. 1.8 G Lie group, $T_eG =: \mathfrak{g}$.

① The diffeomorphism $G \times \mathfrak{g} \to TG$ given by $(g, X) \mapsto L_X(g)$ (resp. $(g, X) \mapsto R_X(g)$) of Prop. 1.7 is called the natural left- (resp. right-) trivialization of $TG \to G$:

$$\begin{tikzcd} TG \arrow[rr, "\sim"] \arrow[dr] & & G \times \mathfrak{g} \arrow[dl, "pr_1"] \\ & G & \end{tikzcd}$$

② For $X \in \mathfrak{g}$, L_X (resp. R_X) is called the left- (resp. right-) invariant vector field generated by $X \in \mathfrak{g}$.

Note that any L_X (resp. R_X) is nowhere vanishing on G and choosing a basis $X_1, \dots, X_n$ of the vector space $\mathfrak{g}$, $L_{X_1}(g), \dots, L_{X_n}(g)$ (resp. $R_{X_1}(g), \dots, R_{X_n}(g)$) form a basis of $T_g G$ $\forall g \in G$.

For any $g \in G$, $\lambda_g^* [\xi, \eta] = [\lambda_g^* \xi, \lambda_g^* \eta] = [\xi, \eta]$ $\forall \xi, \eta \in \mathfrak{X}_L(G)$.

By Prop. 1.7, $\mathfrak{X}_L(G) \subseteq \mathfrak{X}(G)$ is hence a finite-dimensional subalgebra of $(\mathfrak{X}(G), [\,,\,])$,

Via isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{X}_L(G)$ from Prop. 1.7. 3, we can transport $[\,,\,]$ to a bracket on $\mathfrak{g}$.

<u>Def. 1.8</u> Suppose G is a Lie group. Then the tangent space $\mathfrak{g} := T_e G$ of the identity $e \in G$ together with the map

$$[\,,\,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$
$$[X,Y] := [L_X, L_Y](e)$$

is called the Lie algebra of G.

One has by construction $L_{[X,Y]} = [L_X, L_Y]$.

From the properties of the lie bracket of vector fields it follows:

Prop. 1.10 The bracket $[\,,\,]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ in Def. 1.9 is bilinear and the following properties hold:

(i) It is skew-symmetric: $[X,Y] = -[Y,X] \quad \forall X,Y \in \mathfrak{g}$.

(ii) It satisfies the Jacobi-Identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

$\forall X,Y,Z \in \mathfrak{g}$.

Def. 1.11 ① A **real (resp. complex) Lie algebra** is a real (resp. complex) vector space $\mathfrak{g}$ equipped with a $\mathbb{R}$- (resp. $\mathbb{C}$-) bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ s.t. (i) and (ii) of Prop. 1.10 hold.

② A **Lie algebra homomorphism (resp. isomorphism)** between Lie algebras $\mathfrak{g}$ and g is a linear map $\varphi : \mathfrak{g} \to g$ s.t. $\varphi([X,Y]_{\mathfrak{g}}) = [\varphi(X), \varphi(Y)]_{g}$.

(resp. a linear isomorphism)

③ A **(Lie) subalgebra of** a Lie algebra $\mathfrak{g}$ is a subspace

$\mathfrak{h}$ of $\mathfrak{g}$ s.t. $[X,Y] \in \mathfrak{h}$ $\forall X,Y \in \mathfrak{h}$.

Examples

(1) Consider a finite-dimensional vector space V as a Lie group w.r.t. $+$.

Then the left-trivialization of TV is the usual one $TV = V \times V$ and the left-invariant vector fields correspond to constant functions $V \to V$.

In particular, the Lie bracket of two left-inv. vector fields vanishes.

Hence, the Lie algebra of V is just V equipped with the zero bracket ($[v,w]=0 \quad \forall\, v,w \in V$).

Later, we will see that Lie algebra of any commutative (abelian) Lie group has always zero Lie bracket, i.e. is an abelian Lie algebra (as one says).

(2) G and H two Lie groups with Lie algebra $(\mathfrak{g}, [\,,]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\,,]_{\mathfrak{h}})$.

Then the Lie group $G \times H$ has Lie algebra :

$$T_{(e,e)}(G \times H) = T_e G \times T_e H = \mathfrak{g} \oplus \mathfrak{h}.$$

with the Lie bracket

$$[(X,Y),(X',Y')] = ([X,X']_{\mathfrak{g}}, [Y,Y']_{\mathfrak{h}}).$$

$\forall\, X, X' \in \mathfrak{g}$, $Y, Y' \in \mathfrak{h}$

Hence, the Lie algebra of $G \times H$ is what one calls the direct sum of the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$.
(Check this as an exercise).

(3) $G = GL(n, \mathbb{R}) \subseteq M_n(\mathbb{R})$.

$\mathfrak{g} = \underline{M_n(\mathbb{R})}$ $\qquad\qquad TGL(n,\mathbb{R}) = GL(n,\mathbb{R}) \times M_n(\mathbb{R})$.

$\overset{!!}{\cdot\cdot}$

$\mathfrak{gl}(n, \mathbb{R})$

For $A \in GL(n,\mathbb{R})$, $\lambda_A : GL(n,\mathbb{R}) \to GL(n,\mathbb{R})$

is the restriction of the linear map $\lambda_A : M_n(\mathbb{R}) \to M_n(\mathbb{R})$.

$\Longrightarrow T_B\lambda_A(B, X) = (AB, AX)$.

$\qquad\qquad \in T_B GL(n,\mathbb{R}) \qquad X \in M_n(\mathbb{R})$ and (A, AX) with AX.

$\Longrightarrow L_X(A) = T_{Id}\lambda_A X = AX$ (Identity (Id, X) with X and

Viewing L_X as a function $GL(n,\mathbb{R}) \longrightarrow M_n(\mathbb{R})$, we know that

$$[L_X, L_Y](\mathrm{Id}) = \underline{\overbrace{T_{\mathrm{Id}} L_Y}^{\text{right-mult. by } Y} \overbrace{L_X(\mathrm{Id})}^{X}} - T_{\mathrm{Id}} L_X \overbrace{L_Y(\mathrm{Id})}^{Y}$$

$$= XY - YX \qquad \forall X, Y \in \mathfrak{g} = M_n(\mathbb{R}).$$

Lie algebra of $GL(n,\mathbb{R})$ is $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{R}) = M_n(\mathbb{R})$ equipped with the Lie bracket given by the commutator of matrices.