Last week : $P \to M$ principal G-bundle

We can form associate bundles , $G \times F \to F$ smooth left-action of G on a mfd. F

$$P \times_G F \longrightarrow M$$

fiber bundle with standard fiber F and structure group G.

For $F = \mathbb{V}$ a representation of G , $V := P \times_G \mathbb{V} \longrightarrow M$ is a vector bundle.

- $\Gamma(P \times_G F) \simeq C^\infty(P, F)^G$

Functorial properties of associated bundles:

- Suppose $p : P \longrightarrow M$ principal G-bundle and let $G \times F \to F$ and $G \times \hat{F} \longrightarrow \tilde{F}$ smooth left-actions on mfld. F and $\hat{F}$.

Any smooth map $\Phi : F \longrightarrow \tilde{F}$ that is G-equivariant, i.e. $\Phi(g \cdot f) = g \cdot \Phi(f) \quad \forall g \in G, \forall f \in F$.

induces a morphism of fiber bundles with structure group G covering the identity on M:

$$P[\Phi] : P \times_G F \longrightarrow P \times_G \tilde{F}$$
$$[(u, f)] \longmapsto [u, \Phi(f)]$$

Note that $\mathrm{Id}_P \times \Phi : P \times F \to P \times \tilde{F}$ is G-equivariant

for $(u,f)\cdot g = (u\cdot g, g^{-1}f)$, hence it induces a well-defined

map $P[\Phi]$ and it is smooth, since $q : P \times F \to P \times_G F$ is

a surjective submersion. It is also a morphism of fiber bundles

with structure group G, since locally $P[\Phi]$ has the form

$(x,f) \mapsto (x, \Phi(f))$ and Φ is G-equiv.

We have an induced map

$$\begin{array}{ccc}
\Gamma(P \times_G F) & \xrightarrow{P[\Phi]} & \Gamma(P \times_G \tilde{F}) \\
\wr\wr & & \wr\wr \\
C^\infty(P, F)^G & \longrightarrow & C^\infty(P, \tilde{F})^G \\
h : P \to F & \longmapsto & \Phi \circ h : P \to \tilde{F}
\end{array}$$

In particular, for any morphism $\Phi : \mathbb{V} \to \mathbb{W}$ of G-representations, we get an induced morphism of vector bundles.

$$\begin{tikzcd}
V := P \times_G \mathbb{V} \arrow[rr, "P[\Phi]"] \arrow[dr] & & W := P \times_G \mathbb{W} \arrow[dl] \\
& M &
\end{tikzcd}$$

If Φ is an isomorphism, then so is $P[\Phi]$.

- $p : P \to M$ and $\tilde{p} : \tilde{P} \to \tilde{M}$ principal bundles with structure gr. G and $\tilde{G}$ resp., $\tau : G \to \tilde{G}$ Lie group homomorph. and $\Phi : P \to \tilde{P}$ a morphism of principal bundles over τ covering $\underline{\Phi} : M \to \tilde{M}$.

Given a left action $\underline{\tilde{G} \times F \to F}$, then we get a left action $G \times F \to F$ by $g \cdot f = \tau(g) \cdot f$.

Then we get an induced map:

$$\Phi(F) : P \times_G F \longrightarrow \tilde{P} \times_{\tilde{G}} F$$
$$[u, f] = [\Phi(u), f].$$

This defines a morphism of fiber bundles covering $\underline{\Phi} : M \to \tilde{M}$.
If $F = V$ and $\tilde{G} \times F \to F$ a representation, $\Phi[V]$ is a morphism of vector bundles.

Examples

① $\mathcal{F}(M) \longrightarrow M$ frame bundle of an n-dim. mfd. M.

Any representation $\mathbb{V}$ of $GL(n,\mathbb{R})$ induces a vector bundle

$$V = \mathcal{F}(M) \times_{GL(n,\mathbb{R})} \mathbb{V} \longrightarrow M$$

and isomorphic represent. induce isomorphic vector bundle.

- $\mathbb{V} = \mathbb{R}^n$, $\mathcal{F}(M) \times_{GL(n,\mathbb{R})} \mathbb{R}^n \cong TM$ $\quad \mathcal{F}(M) \times_{GL(n,\mathbb{R})} \mathbb{R}^{n*} \cong T^*M$.
- Tensor bundles corresp. to representations $\bigotimes^p \mathbb{R}^n \otimes \bigotimes^q \mathbb{R}^{n*}$.

Also, if $\mathbb{V}$ is not irreducible and $\mathbb{V} \cong \mathbb{V}_1 \oplus \dots \oplus \mathbb{V}_N$ is the decomposition into irreducible components, then

$$V = \mathcal{F}(M) \times_{GL(n,\mathbb{R})} \mathbb{V} \cong \mathcal{F}(M) \times_{GL(n,\mathbb{R})} \mathbb{V}_1 \oplus \dots \oplus \mathbb{V}_N$$

$$\simeq \underbrace{\mathcal{F}(M) \times_{GL(n,\mathbb{R})} \mathbb{V}_1}_{\parallel} \oplus \dots \oplus \underbrace{\mathcal{F}(M) \times_{GL(n,\mathbb{R})} \mathbb{V}_N}_{\parallel}$$

$$= V_1 \oplus \dots \oplus V_N$$

(2) Suppose M is a mfd. with a G-structure $\left(\underline{P} \hookrightarrow \mathcal{F}(M), \ \underline{G \subseteq GL(n,\mathbb{R})}, \ P \to M \right)$

$$\underline{P \times_G \mathbb{R}^n} \simeq TM .$$

If $G = O(n)$ (i.e. M is equipped with a Riemann-metric),

then $\underline{\mathbb{R}^n} \simeq \mathbb{R}^{n*}$ as $O(n)$-represent.

$v \longmapsto \langle v, - \rangle$ $\quad \langle g.v, w \rangle = \langle v, g^{-1}.w \rangle$

reflecting the fact that $TM \simeq \mathcal{O}(M) \times_{O(n)} \mathbb{R}^n$ and $T^*M \cong \mathcal{O}(M) \times_{O(n)} \mathbb{R}^{n*}$

are isomorphic via g.

$R \in \Gamma(S^2_B(\Lambda^2 T^*M))$ $\qquad \underline{S^2\Lambda^2 \mathbb{R}^{n*}}$

$\longrightarrow$ elements in $\underline{\underline{S^2\Lambda^2 T^*M}}$

$\subseteq \Lambda^2 T^*M \otimes \Lambda^2 T^*M$

$R(\xi, \eta, \zeta, \nu) =$
$\underline{R(\zeta, \nu, \xi, \eta)}$

which satisfy the Bianchi identity.

$$S^2_B \Lambda^2 \mathbb{R}^{n*} = \ydiagram{2,2}_0 \oplus \ydiagram{2}_0 \oplus \mathbb{R}$$

$$\left(\overset{\shortparallel}{\ydiagram{2,2}} \right)$$

$\ydiagram{2,2}_0$ = Weyl curvature + $\ydiagram{2}_0$ = trace-free Ricci-curv. + scalar curvature.

as $O(n)$-represent. this has 3-irreducible components.

$$R \in \Gamma\big(\Lambda^2 T^*M \otimes \mathrm{End}(TM)\big) \qquad \mathbb{R}^{n*} \otimes \mathbb{R}^n$$

III. CONNECTIONS

3.1 Recall from Global Analysis — Affine connections

M n-dim. mfd. , $TM \to M$

Affine connection on M is what we will later call a linear connection on the vectorbundle $TM \to M$.

What is the idea of this concept?

An affine is a device which identifies tangent spaces at nearby point (or more generally a linear connection on a vector bundle is a device that identifies points in nearby fibers).

Why do we care about such a device ?

- $\gamma : I \to M$ smooth curve, $\gamma'(t) \in T_{\gamma(t)}M \; \forall t \in I$.

  Acceleration of γ can only be defined w.r. to an affine connection:

$$\gamma''(0) = \lim_{t\to 0} \frac{\gamma'(t) - \gamma'(0)}{t}$$

makes no sense, since $\gamma'(t) \in T_{\gamma(t)}M$ and $\gamma'(0) \in T_{\gamma(0)}M$ lie in different vector spaces,

- Directional derivative of a vector field $s : M \to TM$ (or a section $s : M \to V$ of any vector bundle) in direction of a vector (field) $\eta_x \in T_x M$?

We have $T\xi : TM \to TTM$ $(Ts : TM \longrightarrow TV)$

but this ignores the vector bundle structure. We would like to view vector fields (or sections of any vector bundle) as vector-valued maps on M and differentiable them as such in direction of vector fields.

If $TM = M \times \mathbb{R}^n \to \mathbb{R}^n$ is trivial $(V = M \times \mathbb{V} \longrightarrow M)$, then ξ can be seen as a map $\underline{\xi : M \to \mathbb{R}^n}$ (or as maps $M \to \mathbb{V}$) and we can define the directional derivative $(D\xi)(x)(\eta)$ $(Ds(x)(\eta))$ of ξ (resp. s) in direction of $\eta \in T(TM)$ as usual:

If $\gamma : I \to M$ C^∞-curve, $\gamma(0) = x$, $\gamma'(0) = \eta_x$, then

$$Ds(x)(\eta_x) = \frac{d}{dt}\Big|_{t=0} s(\gamma(t)) = \lim_{t\to 0} \frac{s(\gamma(t)) - s(\gamma(0))}{t}$$

makes no sense for non-trivial TM.

and $Ds(x) : T_xM \to \mathbb{R}^n$ is a linear map.

Similarly, for $s : M \to V$.

An affine connection (resp. linear connection on any vector bundle $V \to M$) is a choice of identification of the tangent spaces (or the fibres) for nearby points. (It "connects" the fibres at nearby point).

Two (equiv.) view points on affine connection we have already seen,

① Directional derivative perspective

$$\nabla : \Gamma(TM) \longrightarrow \Gamma(T^*M \otimes TM) \quad \text{linear}$$

s.t. $\nabla_\eta f\xi = f\nabla_\eta \xi + df(\eta)\xi \qquad \forall \xi, \eta \in \Gamma(TM), \forall f \in C^\infty(M, \mathbb{R})$

(Leibniz rule)

② Parallel transport: For any C^∞-curve $\gamma : I \to M$, $t_1, t_2 \in I$ one has a linear isomorphism: $Pt_{t_2}^{t_1}(\gamma) : T_{\gamma(t_1)}M \longrightarrow T_{\gamma(t_2)}M$, where $Pt_{t_1}^{t_1}(\gamma) = \mathrm{Id}$ and it satisfies some other properties.

1 $\rightsquigarrow$ 2 : For any vector $\xi_{\gamma(t_1)} \in T_{\gamma(t_1)}M$, $\exists!$ ∇-parallel vector field ξ along γ s.t. $\xi(t_1) = \xi_{\gamma(t_1)}$ and $Pt_{t_2}^{t_2}(\gamma)\left(\xi_{\gamma(t_1)}\right) = \xi(t_2)$.

2 $\rightsquigarrow$ 1 $$\left(\nabla_\eta \xi\right)(x) = \frac{d}{dt}\Big|_{t=0} \left(Pt_0^t\right)^{-1}\left(\xi(\gamma(t))\right)$$ for

$\gamma : I \to M$ C^∞-curve

$\gamma(0) = x$, $\gamma'(0) = \eta_x$.

There is a 3rd equiv. point of view on an affine connection which we will discuss now in the context of linear connections on vector bundles.

3.2 Linear Connections on vector bundles

$\Gamma p : TE \to TM$

Def. 3.1 $p : E \longrightarrow M$ is a fiber bundle with stand. fiber F.

Then the vertical bundle of p is the vector subbundle of $TE \to E$ defined by $Ver(E) := \{ \xi \in TE : T p \xi = 0 \}$

$$= \ker(Tp) \subseteq TE.$$

Its fibers $Ver_u(E) := Ver(E)_u \subseteq T_u E$ are called the vertical subspaces.

Remark. Vector subbundle $D \subseteq TE$ of $TE \to E$ = smooth distribution D on E = subset $D \subseteq TE$ s.t. $D \cap T_u E =: D_u$ is a linear subspace and the restriction of $TE \to E$ to $D \to E$ is a vector bundle.

Note that $\underline{\mathrm{Ver}_u(E) = T_u E_{p(u)}}$ $\quad \left(E_{p(u)} = \{u' \in E : p(u') = p(u)\} \right)$

<ins>Exercise</ins> Check that $\mathrm{Ver}(E) \longrightarrow E$ is a vector subbundle of $TE \to E$ of rank $\mathrm{rk}(\mathrm{Ver}(E)) = \dim(F)$.

Moreover, if $p : E \to M$ is a vector bundle, $E_{p(u)}$ is a vector space and so we may identification:

$$\mathrm{Ver}_u(E) = T_u E_{p(u)} \simeq E_{p(u)} \quad \text{for any } u \in E.$$

Fix $u \in E_{p(u)}$: $\Phi_u : E_{p(u)} \to \mathrm{Ver}_u(E)$ $\quad \phi_u(u') = \frac{d}{dt}\Big|_{t=0} (u + tu')$

is the natural isomorphism above.