**Def. 3.2** Suppose $p : V \longrightarrow M$ is a vector bundle. A a linear connection on $V \to M$ is a linear map:

$$\nabla : \Gamma(V) \longrightarrow \Gamma(T^*M \otimes V) = \Gamma(\mathrm{Hom}(TM, V))$$

$$s \longrightarrow \nabla s : TM \to V$$

$$((\nabla s)(\eta) =: \nabla_\eta s$$

s.t. $\nabla_\eta fs = f \nabla_\eta s + df(\eta) s \quad \forall s \in \Gamma(V), \forall \eta \in \Gamma(TM)$
$(*)$ $\quad \forall f \in C^\infty(M, \mathbb{R})$.

( **Remark**: $\nabla : \Gamma(TM) \times \Gamma(V) \longrightarrow \Gamma(V)$

- $\mathbb{R}$-bilinear $(\eta, s) \longmapsto \nabla_\eta s$
- $C^\infty(M, \mathbb{R})$-linear in $\eta$
- and satisfies $(*)$ in $s$. )

$(*)$ Leibnitz-rule

<u>Example</u> Affine connections on $M$ are linear connections on $TM \to M$.

<u>Thm. 3.3</u> Suppose $p: V \to M$ is a (smooth) vector bundle. Then there exist a linear connection $\nabla$ on $V \to M$ and the space of all linear connections is an affine space modelled on the vector space $\Gamma(T^*M \otimes \mathrm{Hom}(V,V)) = \Gamma(T^*M \otimes V^* \otimes V)$ of 1-forms on $M$ with values in the vector bundle $\mathrm{Hom}(V,V)$.

<u>Proof.</u>

<u>Existence</u>: $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ vector bundle atlas for $V \to M$,

$\phi_\alpha : p^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^m$.

For any $\alpha \in I$ and a section $s \in \Gamma(V)$ with support in $U_\alpha$,

define $\underline{\nabla^\alpha_\xi s}\,(x) := \phi_\alpha^{-1}\Big( (x, (\xi \cdot s_1)(x), \dots, \xi \cdot s_m(x) \Big)$ $\underline{\forall x \in U_\alpha}$

where $\phi_\alpha(s(x)) = (x, s_1(x), \dots, s_m(x))$, and $(\nabla^\alpha_\xi s)(y) = 0$

$\forall y \in M \setminus U_\alpha$.

Now let $\{f_\alpha\}$ be a partition of unity subordinate to the cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$. Then $\nabla_\xi s := \sum_{\alpha \in I} \underline{\nabla^\alpha_\xi (f_\alpha s)}$ for $\xi \in \Gamma(TM), s \in \Gamma(V)$

defines a linear connection on $V \to M$.

✓

<u>Freedom</u>: $A \in \Gamma(T^*M \otimes \mathrm{Hom}(V,V))$

$A : \Gamma(TM) \times \Gamma(V) \to \Gamma(V)$

$C^\infty(M,\mathbb{R})$-bilinear.

$\hat{\nabla}_\xi s := \nabla_\xi s + A(\xi, s)$ is again a linear connection

$$\hat{\nabla}_\xi fs = \nabla_\xi fs + A(\xi, fs) = f\nabla_\xi s + df(\xi)s + fA(\xi,s)$$
$$= f\hat{\nabla}_\xi s + df(\xi)s$$

Conversely, $\hat{\nabla}$ and $\nabla$ are two connections and set

$$A(\xi,s) := \hat{\nabla}_\xi s - \nabla_\xi s .$$

Evidently, $A(f\xi, s) = fA(\xi, s)$ and

$$A(\xi, fs) = \hat{\nabla}_\xi fs - \nabla_\xi fs = f(\hat{\nabla}_\xi s - \nabla_\xi s) + df(\xi)s - df(\xi)s$$
$$= fA(\xi,s) .$$

$\Rightarrow A \in \Gamma(T^*M \otimes \mathrm{Hom}(V,V))$.

$\square$

How do things look in local coordinates?

$$\begin{array}{c} V \\ \downarrow \\ M \end{array}$$

Suppose $(U_\alpha, u_\alpha)$ is a chart for $M$ $(\dim(M) = n)$

and $(U_\alpha, \phi_\alpha)$ is a vector bundle chart for $V \to M$ $(\operatorname{rank}(V) = m)$.

Then $\underline{p^{-1}(U_\alpha)} \xrightarrow{\phi_\alpha} \underline{U_\alpha \times \mathbb{R}^m} \xrightarrow{u_\alpha \times id} u_\alpha(U_\alpha) \times \mathbb{R}^m \subseteq \mathbb{R}^n \times \mathbb{R}^m$

is a chart for $V$.

$$\xi = \sum_{i=1}^{n} \xi^i \frac{\partial}{\partial u^i}, \quad s = \sum_{j=1}^{m} s^j \tilde{e}_j$$

$\tilde{e}_j$ is the local section of $V$ defined on $U_\alpha$ s.t. $\phi_\alpha(\tilde{e}_j(x)) = (x, e_j)$

$e_j$ - standard basis vector in $\mathbb{R}^m$.

$$\nabla_\xi s = \sum_{i=1}^{n} \sum_{j=1}^{m} \xi^i \nabla_{\frac{\partial}{\partial u^i}} \underline{s^j \tilde{e}_j} =$$

$$= \sum_{i,j} \xi^i \underline{\underline{\left(\frac{\partial s^j}{\partial u^i}\right)}} \tilde{e}_j + \sum_{i,j,k} \xi^i \underline{\underline{s^j}} \Gamma^k_{ji} \tilde{e}_k$$

for smooth function $\Gamma_{ji}^{k}$ $(i=1,\dots,n\ ,\ j,k=1,\dots,m)$

characterised by $\nabla_{\frac{\partial}{\partial u^i}} \tilde{e}_j = \sum_{k=1}^{m} \Gamma_{ji}^{k} \tilde{e}_k$ .

More compactly, we also use this notation : $s \in \Gamma(U)$

$s_\alpha : U_\alpha \to \mathbb{R}^m$ $\quad (\phi_\alpha \circ s)(x) = (x, s_\alpha(x)) \ \forall x \in U_\alpha$.

$$(\nabla_\xi s)_\alpha = \underbrace{ds_\alpha(\xi)}_{\substack{\shortparallel \\ \xi \cdot s_\alpha}} + A_\alpha(\xi)(\underline{s_\alpha(x)}) \qquad \underline{A_\alpha \in \Omega^1(U_\alpha, \mathfrak{gl}(m,\mathbb{R}))}$$

$$A_\alpha\left(\frac{\partial}{\partial u^i}\right) =: A_i$$

$$(A_i)_j^k = \underline{\underline{\Gamma_{ji}^{k}}}$$

Similarly, as for affine connections and vector fields we have:

For $\gamma : I \to M$ $C^\infty$-curve, a section of $V$ along $\gamma$ is a section of $\gamma^* V$ and we have an operator $\nabla_{\gamma'} : \Gamma(\gamma^* V) \to \Gamma(\gamma^* V)$.

<u>Prop. 3.4</u> Suppose $V \to M$ is a vector bundle equipped with a linear connection $\nabla$.

① Suppose $\gamma : I \to M$ is a $C^\infty$-curve and $v \in V_{\gamma(t_0)}$, $t_0 \in I$. Then $\exists !$ section $s$ along $\gamma$ s.t. $(\nabla_{\gamma'} s)(t) = 0$ $\forall t \in I$ and $s(t_0) = v$.

② In the setting of ① suppose $[t_0, t_1] \subseteq I$. Then

$$Pt_{t_0}^{t_1}(\gamma) : V_{\gamma(t_0)} \longrightarrow V_{\gamma(t_1)}$$

$$v \longmapsto s(t_1) \qquad (\text{for } s \text{ as in } ①)$$

is a linear isomorphism. It is called the parallel transport along $\gamma$ defined by $\nabla$.

Proof: $\nabla_{\dot\gamma} s$ is well-def. follows as for affine connection.

and also the rest of the proof is as for affine connections

(cf. Global Analysis).

Remark: There is an abstract notion of parallel transport for vector bundles which is equivalent to a linear connection.

The linear connection is recovered from the parallel transport by

$$\nabla_\xi s(x) := \frac{d}{dt}\Big|_{t=0} \left(P_0^t\right)^{-1}\left(s(\gamma(t))\right),$$

where $\gamma: I \to M$ is a smooth curve with $\gamma(0)=x$, $\gamma'(0)=\xi_x$.

Right-hand side just depends on $\gamma'(0)=\xi_x$ and not on the curve $\gamma$. Follows directly from the local coordinate expression of $(\nabla_{\gamma'} s)(t)$.

**Prop. 3.5** Suppose $V \to M$ is a vector bundle equipped with a linear connection $\nabla$.

① For any $x \in M$ and $v \in V_x$, there exist $s \in \Gamma(V)$ with $s(x) = v$ and $(\nabla_{\xi_x} s)(x) = 0 \quad \forall \xi_x \in T_x M$.

② For $v \in V_x$ set $H_v := T_x s\,(T_x M) \subseteq T_v V$ for a choice of section $s$ as in ①.

Then $H_v \subseteq T_v V$ is independent of the choice of section $s$ as in ① and it is a linear compliment to $Ver_v(V)$:

$$T_v V = H_v \oplus Ver_v(V) \quad \forall v \in V.$$

Moreover, $H = \bigcup_{v \in V} H_v \subseteq TV$ is a vector subbundle of $TV \longrightarrow V$, called the horizontal distribution determined by $\nabla$.

Proof.

① Follows from the local coordinate expression of $(\nabla_\xi s)(x)$ which shows that $(\nabla_\xi s)(x)$ depends only on the value and the first derivative of $s$ at $x$.

For $s(x) = v$ $\exists!$ unique 1-jet $j^1_x s$ s.t. $\nabla_{\xi_x} s(x) = 0$.

(2) For all sections $s$ as in (1), $T_x s : T_x M \to T_v V$ is the same and hence $H_v$ is independent of the choice of $s$.

$\underline{p \circ s = \mathrm{Id}_M}$

Since $T_v p$ is the inverse of $T_x s : T_x M \to H_v$, $T_x s$ is a linear isomorphism and $T_v V = H_v \oplus \mathrm{Ver}_v(V)$.

Check that

$$\begin{tikzcd} H \arrow[r, phantom, "\subseteq"] \arrow[dr] & TV \arrow[d] \\ & V \end{tikzcd}$$

is a smooth vector subbundle (Exercise).

**Def. 3.6** $\underline{\underline{V}} \to \underline{\underline{M}}$ vector bundle with a linear connection $\nabla$.

Then for $x \in M$, $v \in V_x$, we have a linear map, called the horizontal lift,

$$\mathrm{Hor}_v : T_x M \longrightarrow H_v \subseteq T_v V$$
$$\xi_x \longmapsto \xi_v^{hor},$$

$T_v V = H_v \oplus Ver_v(v)$

$$\begin{array}{c} T_v V \\ T_v p \downarrow \\ T_x M \end{array}$$

$T_v p : H_v \xrightarrow{\sim} T_x M$

where $\xi_v^{hor}$ is the unique vector in $H_v$ s.t. $T_v p\, \xi_v^{hor} = \xi_x$.

For a vector field $\xi \in \Gamma(TM)$,

$\xi^{hor} : v \longmapsto \xi_v^{hor}$ defines a vector field on $V$, called the horizontal lift of $\xi$ w.r.t. $\nabla$.

__Remark__ : $\mathrm{Hor}_v : T_xM \longrightarrow H_v \subset T_vV$

$$\gamma'(0) \longmapsto \frac{d}{dt}\Big|_{t=0} \underline{Pt_0^t(\gamma)(v)} \in H_v$$

for any $\gamma$ with $\gamma(0) = x$

$\left(Pt_0^t(\gamma) : V_{\gamma(0)} \to V_{\gamma(t)}\right.$

~~forms~~ flow of a vector field $Y(t,v)$ on the total space of $\gamma^*V$) .

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Suppose $\gamma : I \longrightarrow M$ is a smooth curve, then for any $v \in V_{\gamma(0)}$

$\exists!\ \tilde{\gamma}_v : I' \longrightarrow V$ , $I' \subseteq I$ s.t.

$$\begin{tikzcd} & V \arrow[d, "p"] \\ I' \arrow[ur, dashed, "\tilde{\gamma}_v"] \arrow[r, "\gamma"] & M \end{tikzcd}$$

commutes and

$\tilde{\gamma}_v(0) = v$

$\tilde{\gamma}_v'(t) \in H_{\gamma(t)}\ \forall t \in I'$ .

$\tilde{\gamma}_v$ is called the horizontal lift of $\gamma$ w.r. to $\nabla$ and $v$

$$\tilde{\gamma}_v(t) = Pt_0^t(\gamma)(v) \in V_{\gamma(t)}$$

For any $y \in p^{-1}(\gamma(t)) \in V_{\gamma(t)}$, let $\xi_y^{hor}$ be the horizontal lift of $\gamma'(t)$., ~~can be~~ This can be extended to a vector field depending on all of $V$. Then $\tilde{\gamma}_v$ is the unique integral of this vector field with $\tilde{\gamma}_v(0) = v$.