- Lie group G
- its Lie algebra : $\mathfrak{g} = T_e G$ $\quad [\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$.
- $\mathfrak{g} \cong \underline{\mathcal{X}_L(G)}$

Prop. 1.12 Suppose G and H are Lie groups and $\varphi : G \longrightarrow H$ is a Lie group homomorphism.

① Then $\varphi' := T_e\varphi : T_e G = \mathfrak{g} \longrightarrow T_e H = \mathfrak{h}$ $\quad (\varphi(e) = e)$.

is a homomorphism of Lie algebras.

② If G is commutative/abelian, then the Lie bracket on $\mathfrak{g}$ is zero (i.e. $(\mathfrak{g}, [\,,\,])$ is what one calls an abelian Lie algebra).

Proof

① (*) Recall from Global Analysis, If $f: M \longrightarrow N$ C^∞-map between mfds. and $\xi_i \in \mathfrak{X}(M)$ and $\eta_i \in \mathfrak{X}(N)$ for $i = 1,2$ are f-related (i.e. $T_x f \xi_i(x) = \eta_i(f(x))$ $\forall x \in M$), then $[\xi_1, \xi_2]$ is f-related to $[\eta_1, \eta_2]$.

By assumption, $\varphi(gh) = \varphi(g)\varphi(h)$, which says
$\varphi \circ \lambda_g = \lambda_{\varphi(g)} \circ \varphi$ $\quad \forall g \in G$.

Differentiating $\Longrightarrow$ $T_g \varphi \circ T_e \lambda_g = T_e \lambda_{\varphi(g)} \circ \varphi'$

$\Longrightarrow$ evaluating this at $X \in \mathfrak{g}$

$$T_g\varphi\, \underline{L_X}(g) = \underline{T_e\lambda_{\varphi(g)}\varphi'(X)} = L_{\varphi'(X)}(\varphi(g))$$

So, L_X and $L_{\varphi'(X)}$ are φ-related $\forall X \in \mathfrak{g}$.

By $(*)$, for any $X, Y \in \mathfrak{g}$, $[L_X, L_Y]$ is φ-related to $[L_{\varphi'(X)}, L_{\varphi'(Y)}]$, that is,

$$T\varphi\, [L_X, L_Y] = [L_{\varphi'(X)}, L_{\varphi'(Y)}] \circ \varphi \qquad [\varphi'(X), \varphi'(Y)]$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad /\!/$$

Evaluate at $e \in G$: $\underline{\varphi'([X,Y])} = [L_{\varphi'(X)}, L_{\varphi'(Y)}](e)$

② If G is abelian, then

$\nu : G \to G$ is a group homomorphism.

$$\nu(gh) = h^{-1} \cdot g^{-1} = g^{-1} \cdot h^{-1} = \nu(g)\,\nu(h).$$

G abelian

By ①, $\nu' : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homom.

$\nu' = -\mathrm{Id}_{\mathfrak{g}}$

Lemma 1.5

$\implies \forall X, Y \in \mathfrak{g}, \quad \underline{-[X,Y]} = [-X,-Y] = [X,Y]$

$\implies [X,Y] = 0 \quad \forall X, Y \in \mathfrak{g}.$

__Cor. 1.13__ Suppose G is a Lie group and $H \subseteq G$ a Lie subgroup. Then the Lie algebra $\mathfrak{h}$ of H is naturally a subalgebra of $\mathfrak{g}$.

In particular, the Lie algebra of any matrix group $H \subseteq GL(n,\mathbb{R})$ is a subalgebra of $(\mathfrak{gl}(n,\mathbb{R}), [\,,\,])$, where $[\,,\,]$ is the commutator of matrices.

__Proof:__ Apply ① of Prop. 1.12 to the inclusion $i: H \hookrightarrow G$.

(Hence, $T_e i = i' : \mathfrak{h} \hookrightarrow \mathfrak{g}$ is the natural inclusion $T_e H \hookrightarrow T_e G$.)

Prop. 1.14 Suppose G is a Lie group with Lie alg. $(\mathfrak{g}, [\,,\,])$.

① $R_x = \nu^* L_{-x} \quad \forall x \in \mathfrak{g}$

② $[R_x, R_y] = -R_{[x,y]} \quad \forall x, y \in \mathfrak{g}$.

③ $[L_x, R_y] = 0 \quad \forall x, y \in \mathfrak{g}$.

Proof. see homework.

Now let us study the flow of left (resp. right-) invariant vector fields.

Prop. 1.15 G a Lie group, ξ a left (resp. right-) invariant vector field on G.

① $Fl^{\xi}_t(g) = g \cdot Fl^{\xi}_t(e) \quad \forall g \in G$ (resp. $Fl^{\xi}_t(g) = Fl^{\xi}_t(e) \cdot g \quad \forall g \in G$).

② ξ is complete.

Proof: ① If ξ is left-invariant, then ξ is λ_g-related to itself $\forall g \in G$. Hence,

$$Fl_t^\xi \circ \lambda_g = \lambda_g \circ Fl_t^\xi \qquad \forall g \in G.$$

Evaluate at e : $Fl_t^\xi(g) = g \cdot Fl_t^\xi(e)$

(2) Follows from a criteria we proved in Global Analysis and (1) : If $\exists\, \varepsilon > 0$ s.t. any integral curve through any point of a vector field ξ is defined on $(-\varepsilon, \varepsilon)$, then ξ is complete.

Def. 1.16 G Lie group.

A one parameter subgroup of G is a Lie group homomorphism

$$\alpha : (\mathbb{R}, +) \longrightarrow G \qquad \left(\text{i.e. } \alpha : \mathbb{R} \to G \text{ is a } C^\infty\text{-curve s.t. } \alpha(s+t) = \alpha(s) \cdot \alpha(t) \ \forall s,t \in \mathbb{R} \right)$$

In particular, $\alpha(0) = e$.

Lemma 1.17 G Lie group, $\alpha : \mathbb{R} \to G$ C^∞-map with $\alpha(0) = e$, $X \in \mathfrak{g}$. Then the following are equiv.:

1. α is a one parameter subgroup with $\alpha'(0) = X$
2. $\alpha(t) = Fl_t^{L_X}(e)$
3. $\alpha(t) = Fl_t^{R_X}(e)$

Proof.

① $\Longrightarrow$ ②

$$\alpha'(t) = \frac{d}{ds}\Big|_{s=0} \alpha(s+t) = \frac{d}{ds}\Big|_{s=0} \overbrace{\underset{\lambda_{\alpha(t)}(\alpha(s))}{\alpha(t)\cdot\alpha(s)}} = T_e\lambda_{\alpha(t)} X = L_X(\alpha(t))$$

$\Longrightarrow$ α is an integral curve of L_X and since $\alpha(0)=e$, we must have $\alpha(t) = Fl_t^{L_X}(e)$ by uniqueness.

② $\Longrightarrow$ ① $\alpha(t) = Fl_t^{L_X}(e)$ is smooth curve in G with $\alpha(0)=e$ and $\alpha'(0) = L_X(\alpha(0)) = X$.

Since it is a flow (cf. Global Analysis) we have:

$$\underline{\alpha(t+s)} = Fl_{t+s}^{L_X}(e) = Fl_t^{L_X}\big(\underbrace{Fl_s^{L_X}(e)}_{=\alpha(s)}\big) \underset{\text{Proof Prop. 1.15}}{=} \underline{\alpha(s)\,\alpha(t)}$$

By exchanging roles of s and t are proves similarly that

① $\Longleftrightarrow$ ③ .

$\square$

Def. 1.18 G a Lie group with Lie alg. $(\mathfrak{g}, [\,,\,])$.

Then the exponential map of G is given by

$$\exp : \mathfrak{g} \longrightarrow G$$

$$\exp(X) := Fl_1^{L_X}(e)$$

By definition, $\exp(0) = e$

Thm. 1.19 G Lie group with Lie algebra $\mathfrak{g}$, $\exp : \mathfrak{g} \to G$ the exponential map. Then the following holds:

① The map exp is smooth and $T_0 \exp : T_0 \mathfrak{g} \longrightarrow T_e G = \underline{\mathfrak{g}}$ equals $\mathrm{Id}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$.

$$T_0 \mathfrak{g} \text{ is } \underline{\mathfrak{g}}$$

Hence, exp restricts to a diffeomorphism from an open neighbourhood of $0 \in \mathfrak{g}$ in $\mathfrak{g}$ to an open neighbhd. of $e \in G$ in G.

(2) For $X \in \mathfrak{g}$, $g \in G$ we have:

$$Fl_t^{L_X}(g) = g\exp(tX)$$
$$Fl_t^{R_X}(g) = \exp(tX) \cdot g$$

Proof.

We know that $(X, g) \mapsto L_X(g)$ is a smooth map $\mathfrak{g} \times G \longrightarrow TG$ by Prop. 1.7.

Hence, $(X, g) \longmapsto (0, L_X(g))$ is a smooth vector field on $\mathfrak{g} \times G$. Its integral curves are $t \mapsto (X, Fl_t^{L_X}(g))$

and they are smooth.

In particular, $(X,t) \longmapsto (X, \underline{Fl_t^{L_X}(g)})$ is smooth

and hence exp is smooth.

Now, $Fl_t^{L_X}(e) = Fl_1^{L_{tX}}(e) = \exp(tX)$

If $c: I \longrightarrow G$ is an integral curve of L_X, then $t \mapsto c(at)$ is an integral curve of $aL_X = L_{aX}$ $\forall a \in \mathbb{R}$.

and so $Fl_t^{L_X}(g) = g \cdot \exp(tX)$ by <u>Prop. 1.15</u>.

For $\quad Fl_t^{R_X}(g) \underset{\uparrow}{=} Fl_t^{R_X}(e)\, g \underset{\uparrow}{=} \exp(tX)\cdot g$

Prop. 1.15 $\qquad\qquad$ Lemma 1.17.

$$T_0 \exp \underline{X} = \frac{d}{dt}\Big|_{t=0} \exp(tX) \overset{②}{=} \frac{d}{dt}\Big|_{t=0} Fl_t^{L_X}(e) = L_X(e) = \underline{X}$$

$\forall X \in \mathfrak{g}$.

$\Longrightarrow T_0 \exp = \mathrm{Id}_{\mathfrak{g}}$. as claimed.

Examples

(1) Consider the commutative Lie group $(\mathbb{R}_{>0}, \cdot)$

Its Lie algebra is $\mathbb{R}$ with trivial Lie bracket.

The left-invariant vector field generated by $x \in \mathbb{R}$ is

$$L_x(a) = ax$$

$$\shortparallel$$

$$T_1 \lambda_a x$$

Integral curve of L_x through $1 \in \mathbb{R}_{>0}$: $\quad L_x(c(t)) = c'(t)$

$$\shortparallel$$

$$c(t)x \qquad c(0) = 1$$

Solution is $c(t) = e^{tx}$

Hence, $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ is the usual exponential map.

② $G = GL(n, \mathbb{R})$ $X \in \mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$

$L_X(A) = AX$ for $A \in GL(n, \mathbb{R})$.

$$\underline{\underset{\displaystyle c(t)X}{\overset{\displaystyle L_X(c(t))}{\parallel}}} = \underline{c'(t)} \qquad \text{and } c(0) = \mathrm{Id}$$

Unique solution is the matrix exponential: $\exp(tX) = \sum_{k=0}^{\infty} \frac{t^k}{k!} X^k$

(If $\|\cdot\|$ is the operator norm on $M_n(\mathbb{R})$, then

$\|X^k\| \leq \|X\|^k$ and so this power series converges

absolutely and uniformly on compact sets).

$\exp(X+Y) \neq \exp(X)\cdot\exp(Y)$ unless X and

Y commute.

Def. 1.20 (Exponential coordinates)

G Lie group with Lie algebra $\mathfrak{g}$, $V \subseteq \mathfrak{g}$ is an open neighborhd. of $0 \in \mathfrak{g}$ s.t. $\exp|_V : V \to \exp(V) =: U$ is a diffeom. onto an open neighborhd U of $e \in G$.

(1) Then $(U, \exp|_V^{-1})$ is a local chart for G with $e \in U$ and $(\lambda_g(U), \lambda_g \circ \exp|_V^{-1})$ a local chart around $g \in G$.

"Canonical coordinates of the first kind".

(2) Choose a basis $\{X_1, \ldots, X_n\}$ of the vector space $\mathfrak{g}$, then $v : \mathbb{R}^n \longrightarrow G$ given by

$$v(t^1, \ldots, t^n) := \exp(t^1 X_1) \ldots \exp(t^n X_n)$$

restricts to a diffeom. from an open neighbhd. V of $0 \in \mathbb{R}^n$ onto an open neighbhd. of $e \in G$.

Indeed, $\frac{\partial v}{\partial t^i}(0) = X_i$ and so $T_0 v(a^1, \ldots a^n)$

$= a^1 X_1 + \ldots + a^n X_n$

$(U, v|_V^{-1} =: u)$ and $(\lambda_g(U), \lambda_g \circ u)$ are local charts around $e \in G$ and $g \in G$.

" Coordinates of the second kind. ".

Cor. 1.21 Let $\varphi : H \to G$ a continuous group homomorphism between Lie groups H and G.
Then φ is smooth.