Homework 1—Differential Geometry

Due date:16.3. 2021

1. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, set

$$I_{p,q} := \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix} \in M_{p+q}(\mathbb{K}) \qquad J_n := \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} \in M_{2n}(\mathbb{K}),$$

and consider the following subgroups of the general linear group $\mathrm{GL}(n, \mathbb{K})$ (resp. $\mathrm{GL}(2n, \mathbb{K})$).

- The **special linear group** given by

$$\mathrm{SL}(n, \mathbb{K}) = \{A \in \mathrm{GL}(n, \mathbb{K}) : \det_{\mathbb{K}}(A) = 1\}.$$

- The **orthogonal** and the **special orthogonal group**

$$\mathrm{O}(n, \mathbb{K}) = \{A \in \mathrm{GL}(n, \mathbb{K}) : A^t = A^{-1}\} \text{ and } \mathrm{SO}(n, \mathbb{K}) = \mathrm{O}(n, \mathbb{K}) \cap \mathrm{SL}(n, \mathbb{K}).$$

  Note that $A \in \mathrm{O}(n, \mathbb{K})$ implies $\det_{\mathbb{K}}(A) = \pm 1$.

- The **(indefinite) orthogonal group of signature** (p, q) with $p + q = n$:

$$\mathrm{O}(p, q) = \{A \in \mathrm{GL}(n, \mathbb{R}) : A^t I_{p,q} A = I_{p,q}\}.$$

- The **(indefinite) special orthogonal group of signature** (p, q) with $p + q = n$:

$$\mathrm{SO}(p, q) = \mathrm{O}(p, q) \cap \mathrm{SL}(n, \mathbb{R}).$$

- The **symplectic group**

$$\mathrm{Sp}(2n, \mathbb{K}) = \{A \in \mathrm{GL}(2n, \mathbb{K}) : A^t J_n A = J_n\}.$$

- The **(indefinite) unitary group of signature** (p, q) with $p + q = n$

$$\mathrm{U}(p, q) = \{A \in \mathrm{GL}(n, \mathbb{C}) : \bar{A}^t I_{p,q} A = I_{p,q}\}.$$

  Note that $A \in \mathrm{U}(p, q)$ implies $|\det_{\mathbb{C}}(A)|^2 = 1$. Here, $\bar{A}$ denotes the conjugate of A.

- The **(indefinite) special unitary group of signature** (p, q) with $p + q = n$:

$$\mathrm{SU}(p, q) = \mathrm{U}(p, q) \cap \mathrm{SL}(n, \mathbb{C}).$$

  For $q = 0$, one also writes $U(n) := \mathrm{U}(n, 0) = \{A \in \mathrm{GL}(n, \mathbb{C}) : \bar{A}^t = A^{-1}\}$ and $\mathrm{SU}(n) := \mathrm{SU}(n, 0)$.

Show that these groups are Lie groups, compute their dimensions and their Lie algebras $\mathfrak{sl}(n, \mathbb{K})$, $\mathfrak{o}(n, \mathbb{K}) = \mathfrak{so}(n, \mathbb{K})$, $\mathfrak{o}(p, q) = \mathfrak{so}(p, q)$, $\mathfrak{u}(p, q)$ and $\mathfrak{su}(p, q)$.

2. Suppose (G, μ, ν, e) is a Lie group with Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$. For $X \in \mathfrak{g}$ denote by L_X and R_X the left- respectively right-invariant vector field on G generated by X. Show that the following holds:

   (a) $R_X = \nu^* L_{-X}$ for all $X \in \mathfrak{g}$

   (b) $[R_X, R_Y] = -R_{[X,Y]}$ for all $X, Y \in \mathfrak{g}$;

   (c) $[L_X, R_Y] = 0$ for all $X, Y \in \mathfrak{g}$.

   As a hint for (a) note that $\nu \circ \rho^g = \lambda_{g^{-1}} \circ \nu$ and for (c) it might help to show that the vector fields $(0, L_X)$ and $(R_Y, 0)$ on $G \times G$ are μ-related to L_X and R_Y respectively.